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SOLVING THE KINEMATICS OF PLANAR MECHANISMS

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ABSTRACT

This paper presents a general method for the analysis of planar mechanisms consisting of rigid links connected by rotational and/or translational joints. After describing the links as vectors in the complex plane, a simple recipe is outlined for formulating a set of polynomial equations which determine the locations of the links when the mechanism is assembled. It is then shown how to reduce this system of equations to a standard eigenvalue problem, or if preferred, a single resultant polynomial. Both input/output problems and tracing-curve equations are treated.

NOMENCLATURE

n Number of links.
ℓ Number of kinematic loops.
M Mobility of a mechanism.
θ_j $e^{i\Theta_j}$, where Θ_j is an angle, in radians.
$\bar{\theta}_j$ $e^{-i\Theta_j}$.
z^* Complex conjugate of z.

1 INTRODUCTION

A large class of planar mechanisms consists of rigid links connected by rotational (pin) joints and translational (sliding) joints. The goal of kinematic analysis is to determine the motion of the links as one or more input links are displaced. In traditional mechanism design, one is often interested in determining the motion of a single output joint as a function of a single input motion. Such designs are called "function generators." For typical robotic applications, the output is the position and orientation of an "end-effector" link and there are multiple input links. In path-generating mechanisms, there is some point of the mechanism whose path in the plane is of particular interest. We will call such a path a "tracing curve," a generalization of the familiar "coupler curve" studied in connection with four-bar linkages. This paper presents a general method which gives the answer to input/output and tracing-curve problems in terms of either a standard eigenvalue problem or a resultant polynomial equation. These results can be used either to analyze the motion of a given mechanism or as a step in the design of a mechanism to produce a desired motion.

To date, analysis has been performed on specific mechanism types. The analysis of four-bars has an extensive literature, dating back to the last century, some notable examples being Roberts (1875), Cayley (1876) and Darboux (1879). Geared five-bars were analyzed by Primrose and Freudenstein (1963), and various six-bar motions were studied by Primrose, et al. (1967). More recently, Innocenti (1994,1995a) has solved the seven-link Assur linkages, which may be applied to provide input/output solutions for any eight-bar linkage. With the exception of geared linkages, the method of this paper applies to all of these, as well as any variant having one or more sliding joints and also for mechanisms beyond eight links.

Our general method relies on a model of the links as vectors in the complex plane. Of the previously mentioned works, both Darboux (1879) and Primrose, et al. (1967) used the complex-vector approach as a starting point, while Roberts (1875) used it in passing. Formulations closer to ours have been presented under the name of "isotropic coordinates" by Bricard (1927), Haarbleicher (1933), and Groenman (1950), all addressing four-bar linkages. The method used here for formulating the initial system of equations is based on Wampler (1996).

The author knows of only one other result close to the generality of the method presented here, namely, Nielsen (1997). He solves the input/output problem for linkages with only rotational joints using the Dixon determinant of sine-cosine polynomials. The method herein is simpler, being based on a Sylvester-type elimination procedure. Our method applies for any combination of rotational and prismatic joints and also applies to tracing curve problems. Nielsen's method has the advantage of using only real arithmetic.

We begin with a general review of the complex vector formulation for planar linkages. Then, we show how the formulation can be applied to solve input/output problems. A slight variation of this procedure leads to a determination of the algebraic equation for a path-generating mechanism.

2 MODELING IN THE COMPLEX PLANE

We begin by reviewing the formulation, via vectors in the complex plane, of kinematic equations for planar mechanisms.

2.1 Rotational Joints

The modeling procedure is especially simple when all joints are rotational, so we begin with this case. It will subsequently shown that prismatic joints lead to the same mathematical form, so that the same solution procedure applies in any case. We present the formulation in the context of a simple example, as the generalization to arbitrary linkages is clear once this is understood.

The geometry of each link is described in a reference position. Suppose, for example, that point A of the link is located at (A_x, A_y) in the Cartesian plane. We may equivalently say that the location of A is given by the complex vector $a = A_x + iA_y$. Now, if as shown in Fig. 1, the link is displaced by a complex-vector translation t and rotation angle Θ, the new position A' of A is $t + \theta a$, where $\theta = e^{i\Theta}$. Using this notation, the position of a point at the end of a series of links pinned together is easily summed. The location of any point in a tree structure of links can be expressed as a sum that is linear in the rotational variables $\theta_1, \theta_2, \ldots, \theta_{n-1}$. The coefficients in these sums are the complex vectors that describe the links in reference position.

To model a general linkage, all that remains is to express the loop closure conditions. Figure 2 shows a Stephenson six-bar mechanism, whose dimensions are given as

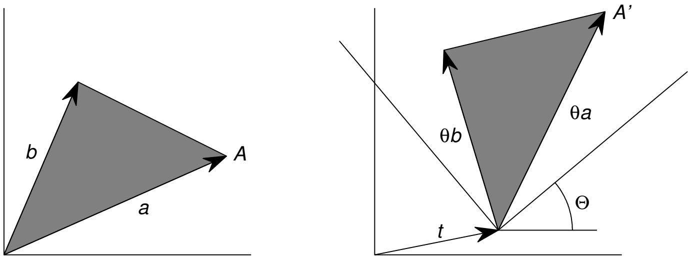

Figure 1. Link in (a) reference position and (b) after translation t and rotation $\theta = e^{i\Theta}$.

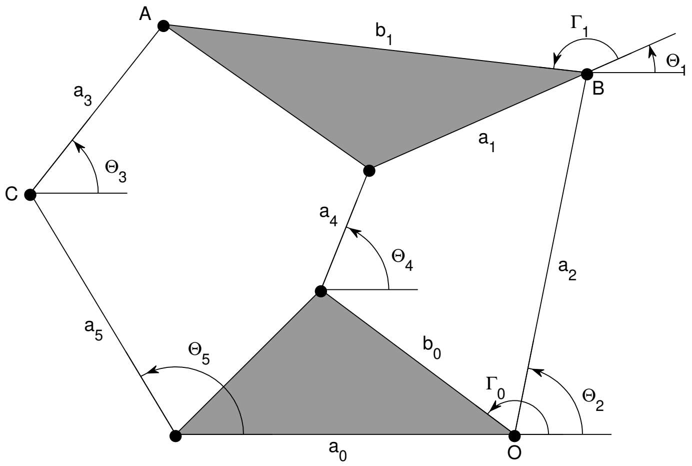

Figure 2. Stephenson six-bar linkage.

lengths $a_0, \ldots, a_5, b_0, b_1$ and angles Γ_0, Γ_1. Equating the vector sums along two paths from O to A, and similarly, from O to B, gives two loop equations. With the conventions $\gamma_j = e^{i\Gamma_j}$ and $\theta_j = e^{i\Theta_j}$, these may be written as

$$\begin{aligned} -a_0 + a_5\theta_5 + a_3\theta_3 &= a_2\theta_2 + b_1\gamma_1\theta_1, \\ b_0\gamma_0 + a_4\theta_4 + a_1\theta_1 &= a_2\theta_2. \end{aligned} \tag{1}$$

These two complex equations completely capture the kinematics of the six-bar linkage. Considering real and imaginary parts, we have the equivalent of 4 real equations in 5 angles, leaving one degree of freedom of motion.

A similar procedure can be applied to any linkage, thus producing one complex equation for each loop closure. Accordingly, the kinematic equations for a general planar mechanism, having ℓ loops and only rotational joints, consist of ℓ equations, each of the form

$$a_0 + \sum_{j=1}^{n-1} a_j \theta_j = 0, \tag{2}$$

where the a_j are complex vectors describing the links in their reference positions. Considering real and imaginary parts of these equations, the mobility of the linkage is thus seen to be $M = (n-1) - 2\ell$, as is well-known.

3 INPUT/OUTPUT SOLUTION

Suppose that M joints are taken as known inputs, thus removing any freedom of motion of the remaining links. Their locations are not unique, however, owing to the nonlinear relation between θ_j and the angles Θ_j. We wish to find all solutions of the closure equations. We will do this by selecting some joint as the primary output joint and eliminating all other joint variables from the closure equations, leaving only a single polynomial equation, frequently called the *input/output equation*. Our method also provides a procedure for back-solving to obtain the values of all the joints in the linkage. We number the joints so that the inputs are Θ_j, $j = n-M, \ldots, n-1$, leaving as outputs the joints Θ_j, $j = 1, \ldots, 2\ell$.

3.1 Isotropic Coordinates

The usual approach to equations of the form we are considering is to take their real and imaginary parts, in which case θ_j becomes $\cos\Theta_j + i\sin\Theta_j$. This system in turn may be converted from trigonometric to algebraic either by a tangent half-angle substitution or by using the unit circle identity relation between sine and cosine.

We take an alternative approach based on the exponential variables θ_j and their conjugates $\bar{\theta}_j$. These are known as *isotropic coordinates*. A more complete discussion of these can be found in Wampler (1996).

We generate equations in the conjugate variables $\bar{\theta}_j$ by simply taking the complex conjugates of the closure equations Eq.(2). This gives ℓ equations of the form

$$a_0^* + \sum_{j=1}^{n-1} a_j^* \bar{\theta}_j = 0, \tag{3}$$

where $*$ indicates conjugation.

We now may treat θ_j and $\bar{\theta}_j$ as independent variables with the constraint that for all rotational joints

$$\theta_j \bar{\theta}_j = 1, \qquad j = 1, \ldots, 2\ell. \tag{4}$$

Altogether, Eqs.(2,3,4) form 4ℓ polynomial equations in the 4ℓ variables $(\theta_j, \bar{\theta}_j)$, $j = 1, \ldots, 2\ell$. The remaining $(\theta_j, \bar{\theta}_j)$, $j > 2\ell$ are all known inputs.

3.2 Reduction to bilinear quadratics

The first step of reduction is to use the linear closure equations to eliminate half of the variables. Unless the mechanism is degenerate, the closure equations (2) can be solved to express ℓ of the variables θ_j as linear combinations of the others. Let us renumber them so that θ_j, $j = \ell+1, \ldots, 2\ell$ are solved in terms of θ_j, $j = 1, \ldots, \ell$. Do the same for the conjugate variables $\bar{\theta}_j$ using Eq.(3). Their solution will be the element-by-element conjugation of the relations for the θ_j. Substitution into the unit vector equations (4) for $j = \ell+1, \ldots, 2\ell$ leaves the following system of 2ℓ equations. For $j = 1, \ldots, \ell$,

$$\theta_j \bar{\theta}_j = 1, \tag{5}$$

$$(b_{0j} + \sum_{k=1}^{\ell} b_{kj}\theta_k)(b_{0j}^* + \sum_{k=1}^{\ell} b_{kj}^* \bar{\theta}_k) = 1. \tag{6}$$

3.3 Solution of Bilinear Equations

It is significant that the equations to be solved are all bilinear. That is, if we divide the variables into two groups along the natural line as $\{\theta_1, \ldots, \theta_\ell\}$ and $\{\bar{\theta}_1, \ldots, \bar{\theta}_\ell\}$, then Eqs.(5,6) are linear in each of the groups when considered separately. Multilinear systems, including bilinear systems, are one of several types for which the resultant can be expressed as a Sylvester-type determinant formula (Sturmfels and Zelevinsky 1994). Separately, in the kinematics literature, Innocenti (1995b) has given an explicit construction for 6 bilinear equations in $3+3$ variables enroute to solving a spatial Burmester problem. In the following paragraphs, we give a procedure which applies to general systems of the form of Eqs.(5,6). Rather than following the prescriptions of the aforementioned references, which apply to general bilinear equations, we take advantage of the especially simple form of Eqs.(5). This reduces the size of the matrices generated to approximately half that required for general bilinear systems.

We will develop a determinant formula for the variable $\bar{\theta}_\ell$. (A procedure symmetric to the one described here would produce a formula for θ_ℓ.) By Eq.(5), we may replace $\bar{\theta}_j$ with θ_j^{-1} for $j = 1, \ldots, \ell-1$ to get equations of the following form: for $j = 1, \ldots, \ell$,

$$\theta_\ell \bar{\theta}_\ell - 1 = 0, \tag{7}$$

$$(b_{0j} + \sum_{k=1}^{\ell} b_{kj}\theta_k)(b_{0j}^* + \sum_{k=1}^{\ell-1} b_{kj}^* \theta_k^{-1} + b_{\ell j}^* \bar{\theta}_\ell) = 1. \tag{8}$$

Consider this system as $\ell+1$ equations in the ℓ unknowns $\theta_1,\ldots,\theta_\ell$ with coefficients that depend on $\bar\theta_\ell$. We seek the values of $\bar\theta_\ell$ which make this overdetermined system consistent.

A Sylvester-type eliminant is obtained by multiplying each of these equations by each of the monomials of degree $\le \ell-1$ in $\theta_1,\ldots,\theta_\ell$. There are $\binom{2\ell-1}{\ell}$ such monomials, yielding $(\ell+1)\binom{2\ell-1}{\ell}$ equations in the expanded set. Keeping $\bar\theta_\ell$ suppressed into the coefficients, the monomials which appear in the expanded set of equations are all monomials of degree $\le \ell$ in θ_j $(j=1,\ldots,\ell)$, along with monomials of degree -1 in θ_k and degree $\le \ell$ in θ_j for $k=1,\ldots,\ell-1$, $j=1,\ldots,\ell$, $j\ne k$. The number of monomials is $\binom{2\ell}{\ell}$ and $(\ell-1)\binom{2\ell-1}{\ell}$, respectively, the total of which is exactly equal to the number of equations in the expanded set. Accordingly, we may assemble all the monomials into a column vector $\mathbf{m}$ of length $(\ell+1)\binom{2\ell-1}{\ell}$ and write the expanded set of equations in matrix form as

$$Q\mathbf{m} = (Q_1 + Q_2\bar\theta_\ell)\mathbf{m} = 0. \tag{9}$$

where Q_1 and Q_2 are square with sparse complex entries. Only the monomials in $\mathbf{m}$ that have no negative exponents have coefficients in which $\bar\theta_\ell$ appears, hence all but $\binom{2\ell}{\ell}$ of the columns of Q_2 are zero. The condition for Eq.(9) to have a nontrivial solution is

$$\det(Q_1 + Q_2\bar\theta_\ell) = 0, \tag{10}$$

which is a polynomial in $\bar\theta_\ell$ of degree $\binom{2\ell}{\ell}$. If this determinant is not identically zero, then it is the resultant of the system.

For general four-bar and six-bar input/output problems $(\ell=1,2)$, Eq.(10) is nontrivial. However, for eight-bar input/output problems $(\ell=3)$, numerical tests show that the determinant is zero. Nevertheless, when the generalized eigenvalue problem of Eq.(9) is solved (using the `EIG` command in Matlab), the correct solutions are found. The number of solutions having non-zero values in all elements of the eigenvector is equal to the known number of solutions to the input/output problem (14, 16, or 18, depending on the linkage type).

It is possible to reduce the size of the eigenvalue problem to no greater than $\binom{2\ell}{\ell}$ by standard methods from linear algebra. Begin by writing the equations in block matrix form as

$$\begin{pmatrix} A_1 & A_2 + B\bar\theta_\ell \end{pmatrix}\begin{pmatrix}\mathbf{m}_1\\ \mathbf{m}_2\end{pmatrix} = 0,$$

which makes explicit the fact that the leading columns do not depend on $\bar\theta_\ell$. Apply Gaussian elimination, with row pivoting, to reduce A_1 to row-echelon form. This produces a system of the form

$$\begin{pmatrix} A_{11} & A_{12}+B_1\bar\theta_\ell \\ 0 & A_{22}+B_2\bar\theta_\ell\end{pmatrix}\begin{pmatrix}\mathbf{m}_1\\ \mathbf{m}_2\end{pmatrix} = 0.$$

If A_1 is full-rank, then the lower row of the block-matrix equation is a square system. Otherwise, one may square it up by taking the first $\binom{2\ell}{\ell}$ rows of $A_{22}+B_2\bar\theta_\ell$ and adding random, linear combinations of the remaining rows. Either way, one ends up with a square system to solve as a generalized eigenvalue problem. Although effective on test problems, this is not put forward as the optimal numerical approach to solving Eq. (9). For greatest efficiency and numerical stability, one should use sparse QR-decomposition techniques. However, we leave a full exploration of this matter to future work.

Finally, the solutions must be tested for physical validity. This is analogous to the more familiar situation of equations with real coefficients in which the physically meaningful solutions must be pure real (zero imaginary part). In our case, we require $\theta_j\theta_j^* = 1$, that is, the rotation vectors must have unit magnitude.

3.4 Prismatic joints

For simplicity, the foregoing discussion was limited to the case of rotational joints. The generalization to prismatic joints requires only a few minor adjustments in formulation.

Referring to Figure 3, suppose that link 2 slides a distance λ along unit vector v, fixed in link 1. Additionally, link 1 is subject to rotation θ_1. Then, then the vector from O to P, expressed in fixed coordinates, is

$$\vec{OP} = \theta_1 a + \lambda\theta_1 v + \theta_1 b.$$

It is convenient to make the substitution $s_2 = \lambda\theta_1$, so that we have a linear expression

$$\vec{OP} = \theta_1 a + s_2 v + \theta_1 b.$$

In this fashion, the loop closure equations for linkages containing prismatic joints remains of the same form as Eq.(2), except that $s_j = \lambda\theta_{j-1}$ appears instead of θ_j when joint j is of prismatic type. In addition, we define $\bar s_j = \lambda\bar\theta_{j-1}$ which then appears instead of $\bar\theta_j$ in the conjugate closure equations (3). If more than one prismatic joint appears in

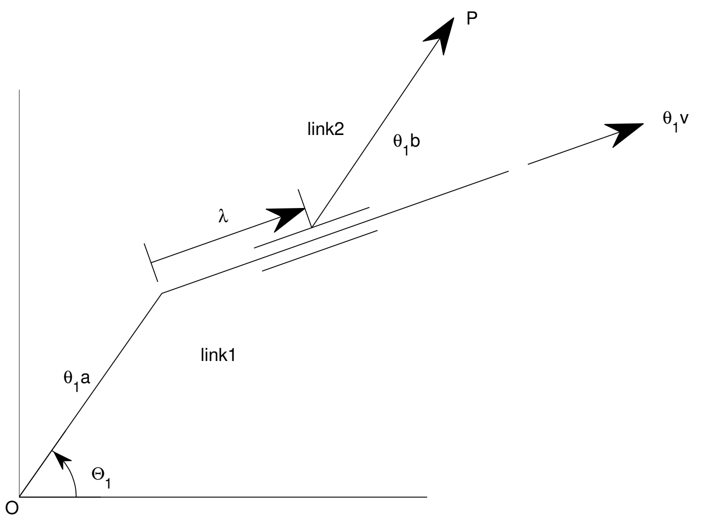

Figure 3. Formulation for prismatic joints.

series, the angle involved in the definitions of s_j and $\bar{s}_j$ will be that of the last previous rotational joint.

We also must develop an equation for prismatic joints to correspond to the unit vector equation (4) that applies in the rotational case. From their definitions in the previous paragraph, it is seen that

$$\bar{\theta}_{j-1}s_j - \theta_{j-1}\bar{s}_j = 0. \qquad (11)$$

In the case that the prismatic joint j connects to the ground link, this simplifies to

$$s_j = \bar{s}_j.$$

The bilinearity of these new relations ensures that a Sylvester-type resultant still exists. If the loop equations are used to eliminate all the prismatic joint variables, the result is a system of the same form as Eqs.(7,8), and one proceeds exactly as before. If instead, some of the prismatic joint variables are retained, then the corresponding identites Eq.(11) must also be retained. Then, as before, one expands the set of equations by multiplying the loop closure equations by monomials of degree $\leq \ell - 1$ in $\mathbf{x}$, where x_j is either θ_j or s_j, depending on joint type. This will lead to a square system of the same form as derived for the rotational case, although of larger dimension. In this fashion, the methodology extends to any number of prismatic joints.

4 TRACING CURVES

In some applications, one seeks the equation of the curve swept out by a specified point of one of the links. We assume that the mobility of the mechanism is 1, hence $n - 1 = 2\ell + 1$ is the number of moving links. For four-bar mechanisms, the point of interest is invariably on the coupler link, as points on the other two moving links merely follow circles. Hence, the point of interest is called the *coupler point*, and its locus is the *coupler curve*. When this is generalized to more complex linkages, the term “coupler link” no longer applies so readily, so we use the terms *tracing point* and *tracing curve*.

Let p be the complex vector giving the absolute location of the tracing point. Then, in addition to the ℓ loop closure equations and their conjugates, one has the complex conjugate equations for p and $\bar{p}$ of the form

$$p = a_0 + \sum_{j=1}^{n-1} a_j\theta_j, \qquad \bar{p} = a_0^* + \sum_{j=1}^{n-1} a_j^*\bar{\theta}_j$$

where, as before, the a_i are complex vectors describing the links in reference position. Additionally, we have the unit vector conditions

$$\theta_j\bar{\theta}_j = 1, \qquad j = 1, \ldots, 2\ell + 1.$$

As in the input/output case, we first reduce the system to bilinear quadratics. We have enough linear relations to eliminate $\theta_j, \bar{\theta}_j$ for $j > \ell$, thereby obtaining $\ell + 1$ bilinear equations. These may be written as, for $j = 1, \ldots, \ell + 1$,

$$(b_{0j} + \sum_{k=1}^{\ell} b_{kj}\theta_k + b_{(\ell+1)j}p)(b_{0j}^* + \sum_{k=1}^{\ell} b_{kj}^*\theta_k^{-1} + b_{(\ell+1)j}^*\bar{p}) = 1. \qquad (12)$$

We have $\ell + 1$ equations in ℓ angles and $p, \bar{p}$, indicating 1 degree of freedom of motion. We wish to eliminate the angle variables to obtain a single tracing curve equation involving only $p, \bar{p}$.

Elimination may be accomplished by multiplying each of Eqs.(12) by the monomials of degree $\leq \ell$ in $\theta_1, \ldots, \theta_\ell$ of which there are $\binom{2\ell}{\ell}$. One finds that this yields a square system of size $(\ell+1)\binom{2\ell}{\ell}$. This may be treated by the same techniques outlined above for the input/output problem. The final system of equations has the form

$$(B_0 + B_1 p + B_2\bar{p} + B_3 p\bar{p})\mathbf{m} = 0, \qquad (13)$$

or

$$\det(B_0 + B_1 p + B_2\bar{p} + B_3 p\bar{p}) = 0, \qquad (14)$$

where the B_i matrices are sparse with complex entries. In particular, between B_2 and B_3, only $\binom{2\ell+1}{\ell}$ monomials in $\mathbf{m}$ have coefficients involving $\bar{p}$. This means that the maximum possible bidegree[1] of the tracing curve with respect to $\bar{p}$ is $\binom{2\ell+1}{\ell}$. The variable p appears in more than this number of columns, but the symmetry of the original relations implies that the same bound applies to the bidegree with respect to p. This is manifested by a linear dependence between the columns of B_1, a fact that has been verified by numerical experiment. Note that the total degree of the tracing curve is $2\binom{2\ell+1}{\ell} = \binom{2(\ell+1)}{\ell+1}$, which is equal to the degree of a $(\ell+1)$-loop input/output equation. These are upper bounds on the degrees, but the relationship holds as well for the actual degrees, as discussed in Wampler (1996).

The procedure just outlined must be modified slightly to handle linkages having prismatic joints. The necessary adjustments follow closely those previously described for input/output equations, so no further elaboration is necessary here.

5 EXAMPLE

We derive an explicit formulation of the solution for the input/output of the Stephenson 6-bar, illustrated in Fig. 2. Please note that this kind of explicit derivation is not necessary since a general-purpose computer code can now handle all possible mechanisms (with RP joints), but the derivation by hand illustrates the mechanics of the algorithm. We have tested a general-purpose code on this and various 8-bar problems, all to good effect.

Suppose that Θ_5 is the input and Θ_2 is the output angle. The loop closure equations, already given in Eq.(1), can be rearranged to solve for θ_3 and θ_4 as linear functions of θ_1, θ_2 and the quantity

$$c = \vec{OC} = -a_0 + a_5\theta_5,$$

which is known when the input angle Θ_5 is given. The result is

$$\begin{aligned} a_3\theta_3 &= -c + a_2\theta_2 + b_1\gamma_1\theta_1, \\ a_4\theta_4 &= -b_0\gamma_0 + a_2\theta_2 - a_1\theta_1. \end{aligned}$$

We use these and their conjugates to eliminate θ_3 and θ_4 from the unit vector relations $\theta_3\bar{\theta}_3 = 1$ and $\theta_4\bar{\theta}_4 = 1$ to obtain

$$\begin{aligned} (-c + b_1\gamma_1\theta_1 + a_2\theta_2)(-\bar{c} + b_1\bar{\gamma}_1\theta_1^{-1} + a_2\bar{\theta}_2) &= a_3^2 \\ (-b_0\gamma_0 - a_1\theta_1 + a_2\theta_2)(-b_0\bar{\gamma}_0 - a_1\theta_1^{-1} + a_2\bar{\theta}_2) &= a_4^2. \end{aligned}$$

When expanded, these equations may be written in the form, for $j = 1, 2$,

$$f_j = \alpha_{0j} + \alpha_{1j}\theta_1 + \alpha_{2j}\theta_2 + \beta_{1j}\theta_1^{-1} + \beta_{2j}\theta_1^{-1}\theta_2 = 0. \quad (15)$$

The α_{kj} coefficients are linear in $\bar{\theta}_2$ and the β_{kj} coefficients are constants. These are given by:

$$\begin{aligned} &\alpha_{01} = c\bar{c} + b_1^2 - a_3^2 - ca_2\bar{\theta}_2 && \alpha_{02} = b_0^2 + a_1^2 - a_4^2 - b_0\gamma_0 a_2\bar{\theta}_2 \\ &\alpha_{11} = b_1\gamma_1(-\bar{c} + a_2\bar{\theta}_2) && \alpha_{12} = -a_1(-b_0\bar{\gamma}_0 + a_2\bar{\theta}_2) \\ &\alpha_{21} = a_2(-\bar{c} + a_2\bar{\theta}_2) && \alpha_{22} = a_2(-b_0\bar{\gamma}_0 + a_2\bar{\theta}_2) \\ &\beta_{11} = -cb_1\bar{\gamma}_1 && \beta_{12} = b_0\gamma_0 a_1 \\ &\beta_{21} = a_2 b_1\gamma_1 && \beta_{22} = -a_2 a_1 \end{aligned}$$

In addition, we have the unit vector equation

$$f_3 = \theta_2\bar{\theta}_2 - 1 = 0. \quad (16)$$

To obtain the resultant for $\bar{\theta}_2$, we multiply each of the three equations Eqs.(15,16) by the monomials 1, θ_1, and θ_2. This produces 9 equations in 9 monomials of the form of Eq.(9). The 9×9 matrix Q is as follows, where the rows appear in the order $f_1, \theta_1 f_1, \theta_2 f_1, f_2, \theta_1 f_2, \theta_2 f_2, f_3, \theta_1 f_3, \theta_2 f_3$, and the columns are labeled by the monomials that appear after the multiplication

$$\begin{array}{c} \begin{array}{ccccccccc} 1 & \theta_1 & \theta_2 & \theta_1^2 & \theta_1\theta_2 & \theta_2^2 & \theta_1^{-1} & \theta_1^{-1}\theta_2 & \theta_1^{-1}\theta_2^2 \end{array} \\ \begin{pmatrix} \alpha_{01} & \alpha_{11} & \alpha_{21} & 0 & 0 & 0 & \beta_{11} & \beta_{21} & 0 \\ \beta_{11} & \alpha_{01} & \beta_{21} & \alpha_{11} & \alpha_{21} & 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha_{01} & 0 & \alpha_{11} & \alpha_{21} & 0 & \beta_{11} & \beta_{21} \\ \alpha_{02} & \alpha_{12} & \alpha_{22} & 0 & 0 & 0 & \beta_{12} & \beta_{22} & 0 \\ \beta_{12} & \alpha_{02} & \beta_{22} & \alpha_{12} & \alpha_{22} & 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha_{02} & 0 & \alpha_{12} & \alpha_{22} & 0 & \beta_{12} & \beta_{22} \\ -1 & 0 & \bar{\theta}_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & \bar{\theta}_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & \bar{\theta}_2 & 0 & 0 & 0 \end{pmatrix} \end{array}.$$

Only the first 6 columns depend on $\bar{\theta}_2$, so the determinant gives a resultant equation of degree 6, which is known to be the correct degree for this linkage. As outlined in Section 3.3, sparse Gaussian elimination can be used to reduce the problem to that of finding the eigenvalues of a 6×6 matrix.

6 CONCLUSIONS

We have given an elimination procedure for both input/output equations and tracing-curve equations. It applies to any planar linkage having rotational or prismatic

[1] Bidegree with respect to $\bar{p}$ is the degree of the polynomial when p is considered a constant.

joints. After forming the eliminant, the problem is most effectively solved as a generalized eigenvalue problem. For input/output problems, the initial size of the resultant matrix is $\frac{1}{2}(\ell+1)\binom{2\ell}{\ell}$, but the problem can be readily reduced to size $\binom{2\ell}{\ell}$ using sparse Gaussian elimination. The joint solutions are found using an off-the-shelf routine for generalized eigenvalue problems.

Formulation of kinematic equations in the complex plane via isotropic coordinates has been found to produce equations in an advantageously simple form; in particular, the loop equations are bilinear quadratics instead the general quadratics that arise in a more typical formulation in the Cartesian plane. This new form of the equations allows the resultant to be expressed as a Sylvester-type determinant.

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