TUTORIAL: State-of-the-Art Flow Field Analysis and Visualization

Vector Field Topology in Flow Analysis and Visualization

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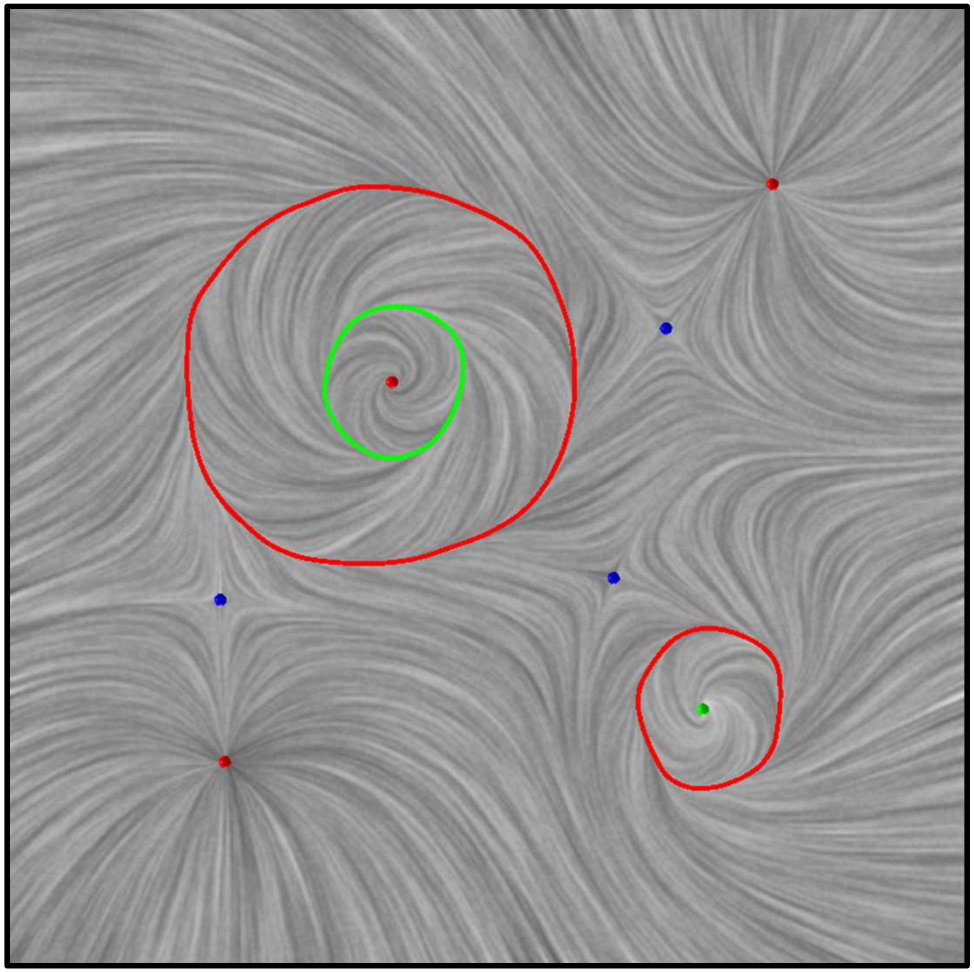
Outline

- Background – why topology?
- What is vector field topology (for steady field)?
- What are the existing variations (i.e., different representations and computations) of topology for steady vector fields?
- Where are we heading?

What Are We Looking For From Flow Data?

- For steady flow



What Are We Looking For From Flow Data?

- For steady flow

Fixed points $V(x_0) = 0$

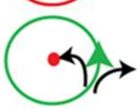
$\varphi(t, x_0) = x$ for all $t \in R$

- Sink
- Source
- Saddle

Periodic orbits

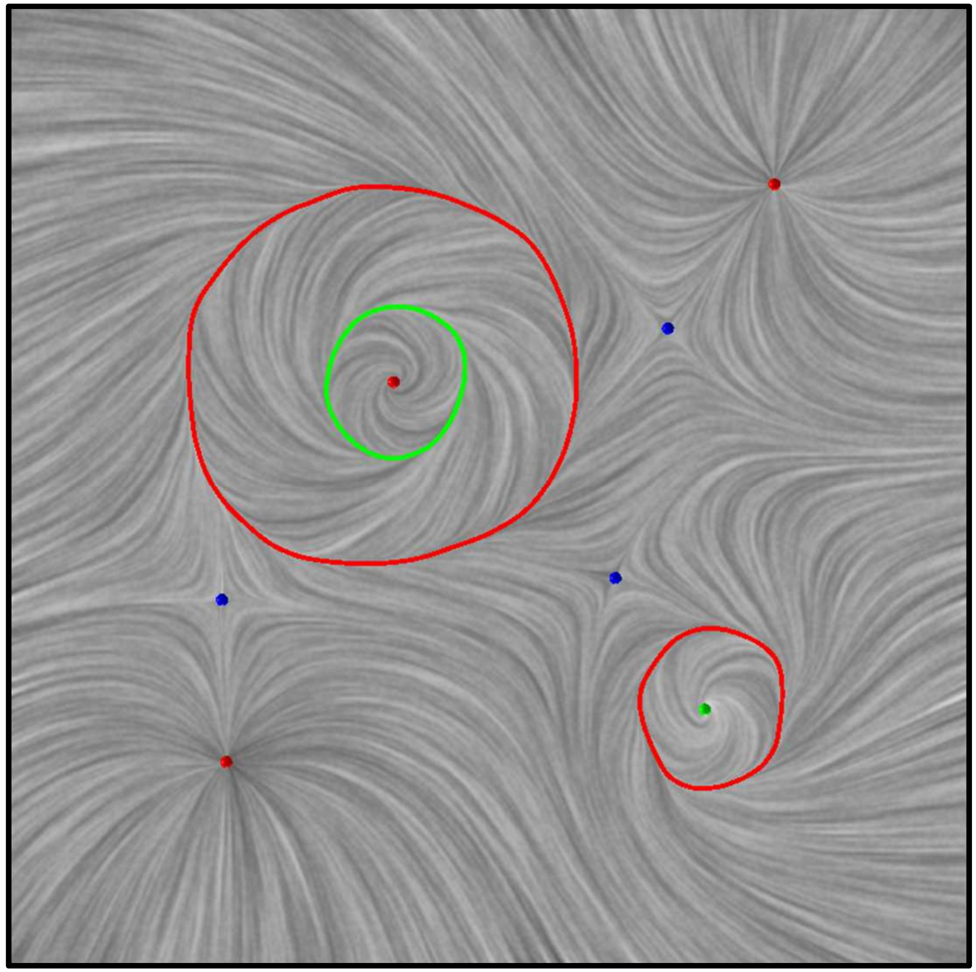
$\exists T_0 > 0$ such that $\varphi(T_0, x) = x$

Attracting



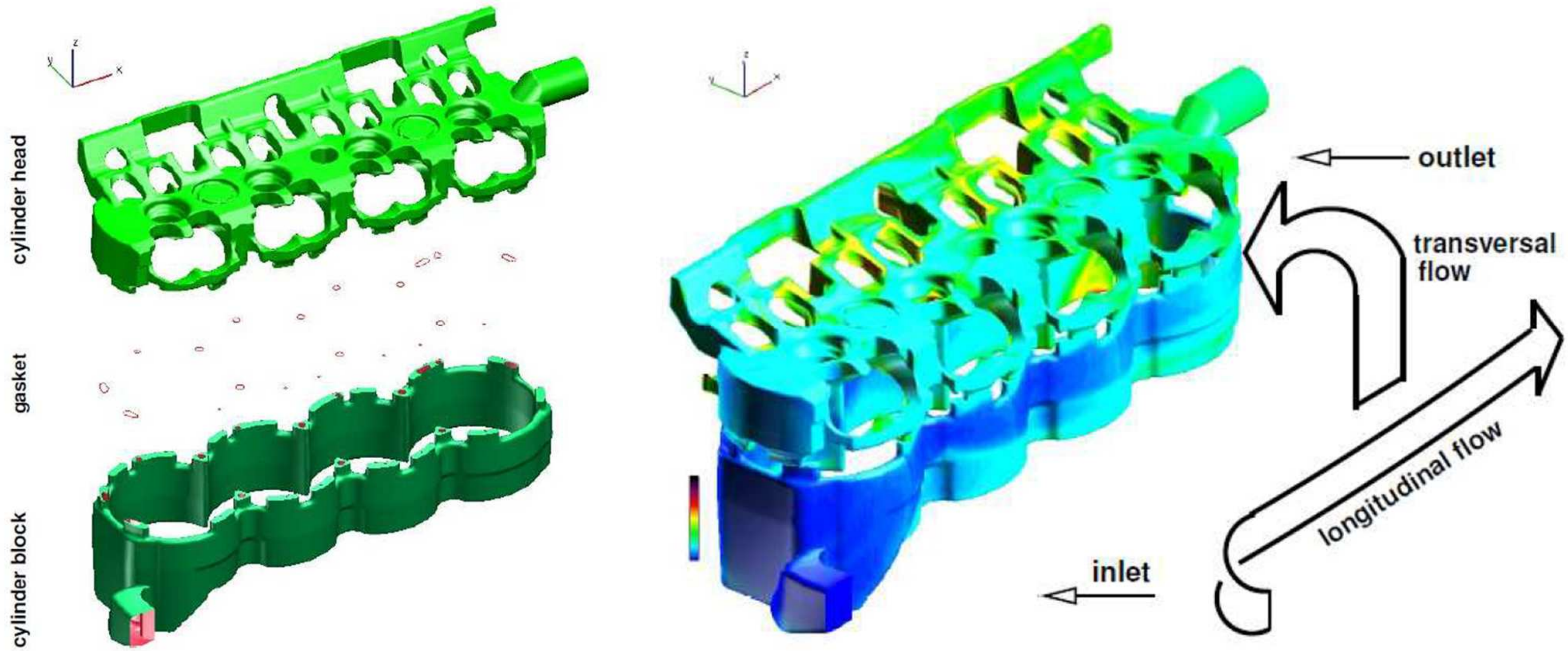
Repelling

They are **flow recurrent** dynamics that trap flow particles forever



Example Application in Automatic Design

- CFD simulation on cooling jacket
- Velocity extrapolated to the boundary



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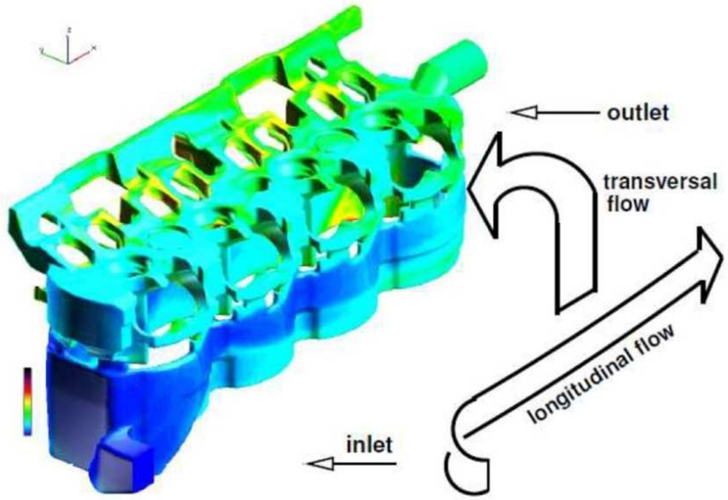
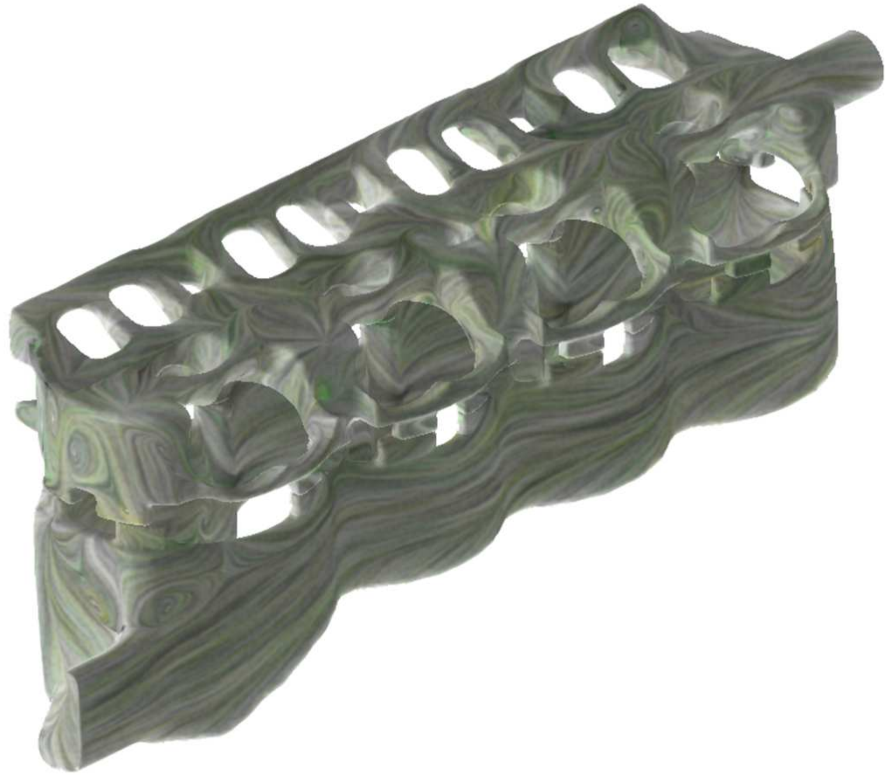
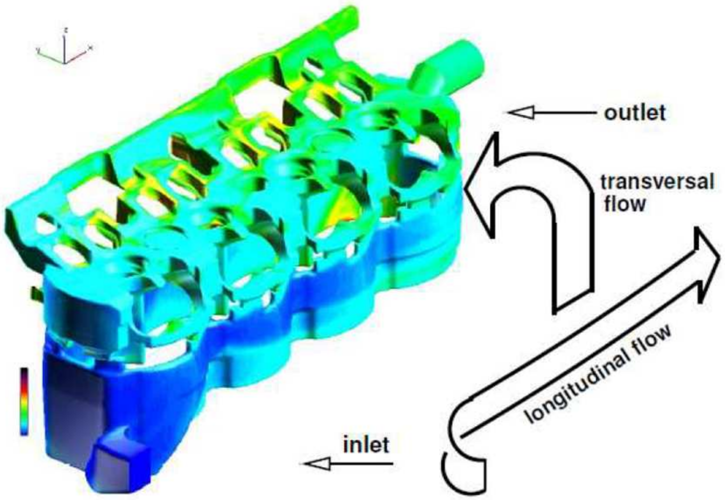


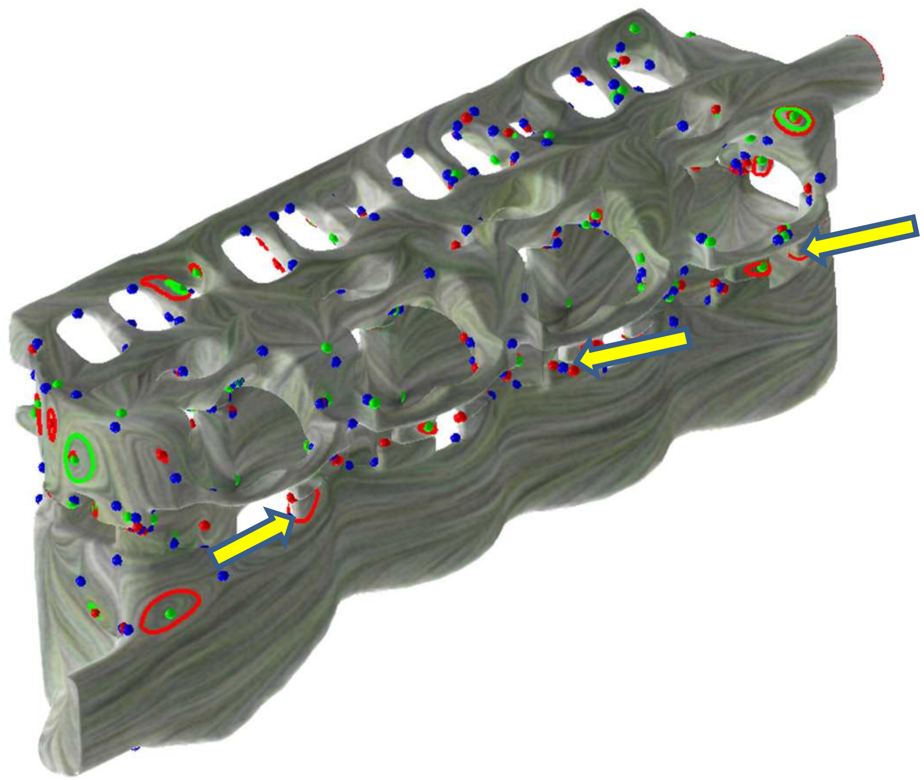


Where are the critical dynamics of interests?

Topology can help!

- CFD simulation on cooling jacket
- Velocity extrapolated to the boundary

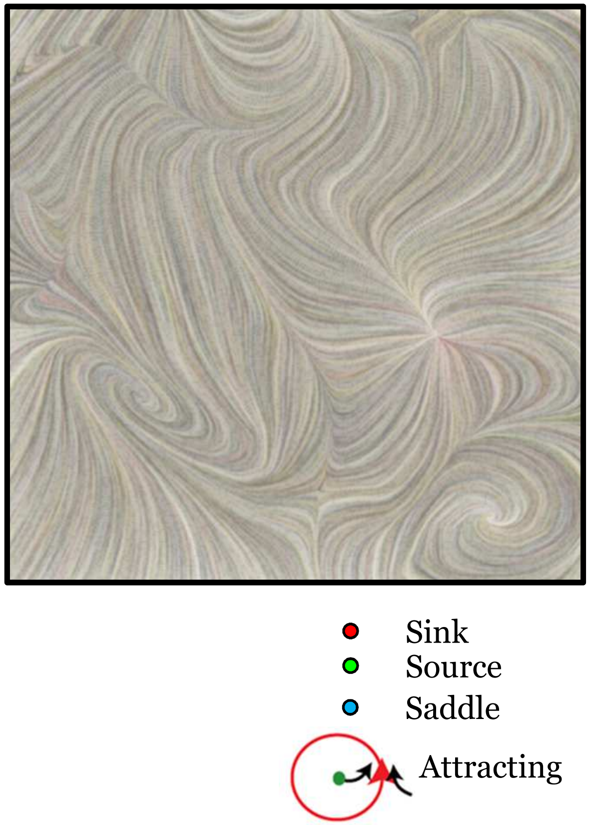




These critical dynamics are parts of vector field topology!

The connections of these (hyperbolic) flow recurrent features give rise to vector field topology!

- It condenses the whole flow information into its skeletal representation or structure, which is sparse.
- It provides a domain partitioning strategy which decomposes the flow domain into sub-regions. Within each sub-region, the flow behavior is homogeneous.
- It is one of those few rigorous descriptors of flow dynamics that are parameter free.
- It defines rigorous neighboring relations between features such that a hierarchy of the flow structure can be derived based on certain importance metric.
- *<u>This is what we need for large-scale data analysis in order to achieve multiscale/level-of-detail exploration!</u>*



Vector Field Topology

- Differential topology
  - Topological skeleton [Helman and Hesselink 1989; CGA91] [Scheuermann et al. Vis97, TVCG98][Tricoche et al. Vis01, VisSym01] [Theisel et al. CGF03][Polthier and Preuss 2003][Weinkauf et al VisSym04] [Weinkauf et al. Vis05] [Chen et al. TVCG07]

- Discrete topology
  - Morse decomposition [Conley 78] [Chen et al. TVCG08, TVCG12]
  - PC Morse decomposition [Szymczak EuroVis11] [Szymaczak and Zhang TVCG12] [Szymczak and Sipeki, Vis13]

- Combinatorial topology
  - Combinatorial vector field [Forman 98]
  - Combinatorial 2D vector field topology [Reininghaus et al. TopoInVis09, TVCG11]

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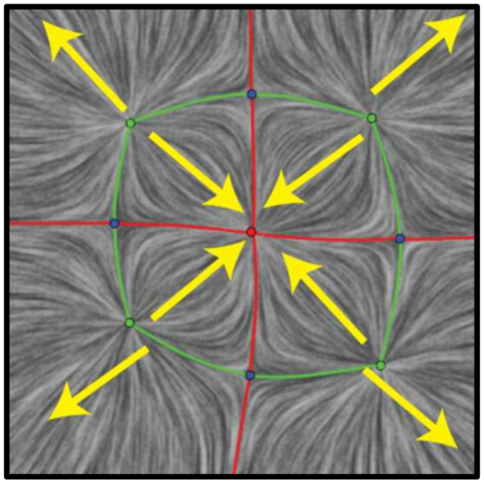

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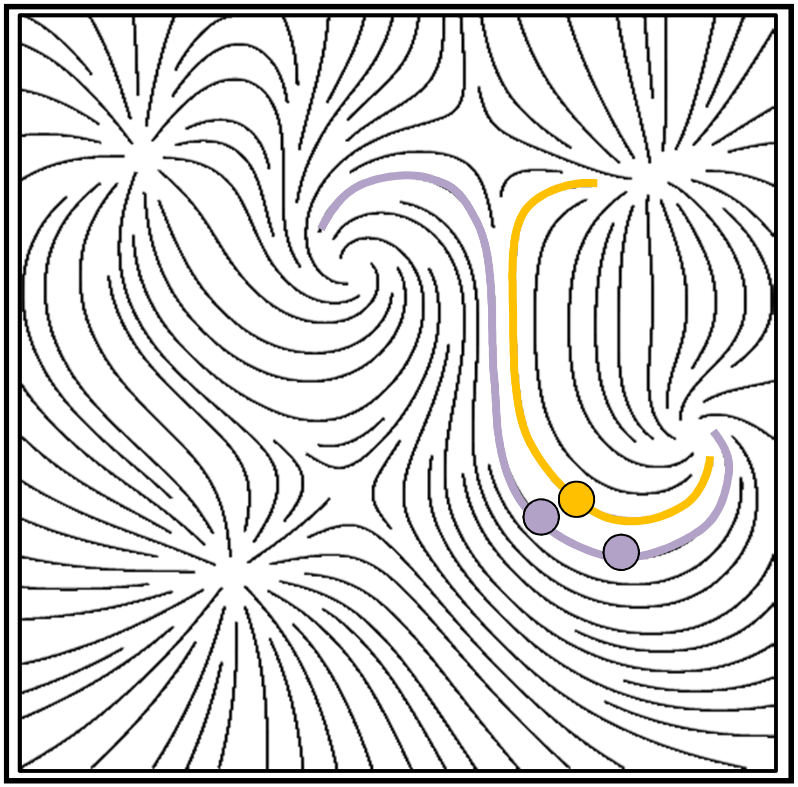
Vector Fields (Recall)

- A **vector field**
  - is a continuous vector-valued function $V(x)$ on a manifold X
  - can be expressed as a system of ODE $\dot{x} = V(x)$
  - introduces a **flow** $\varphi : R \times X \rightarrow X$

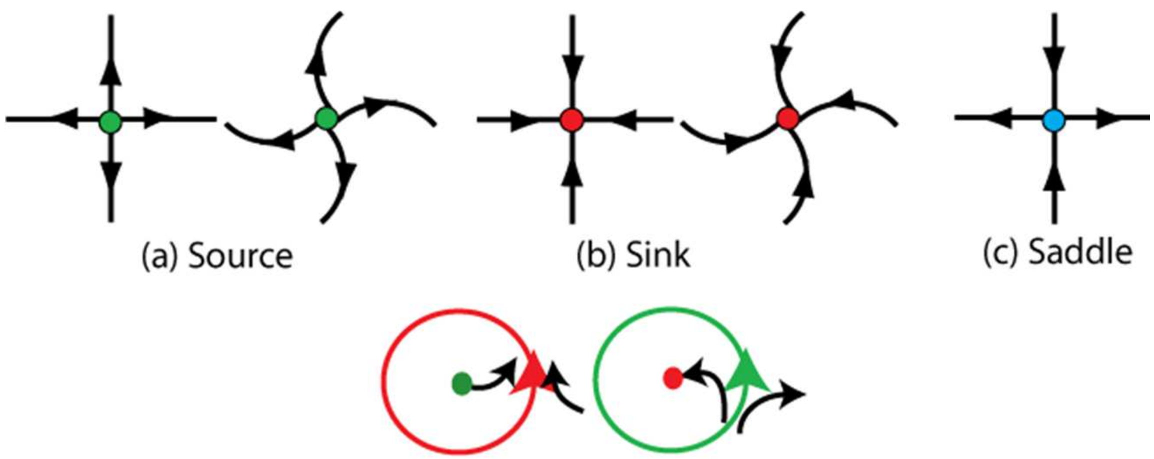
Trajectories

- A **trajectory** of $x \in X$ is $\cup_{t \in R} \varphi(t, x)$
- Given an initial condition, there is a unique solution
  $\mathbf{x}(t) = \mathbf{x}_0 + \int_{0 \le u \le t} \mathbf{v}(\mathbf{x}(u))\, du$
  $\varphi(t_0) = \mathbf{x}_0$
- Uniqueness
- Under time-independent setting a trajectory is also called **streamline**.



Fixed Points and Periodic Orbits

- A point $x \in X$ is a **fixed point** if $\varphi(t, x) = x$ for all $t \in \boldsymbol{R}$
- x is a periodic point if there exists a $T > 0$ such that $\varphi(T, x) = x$. The trajectory of a periodic point is called a **periodic orbit**.



(a) Source (b) Sink (c) Saddle

(d) Periodic orbits

Limit Sets

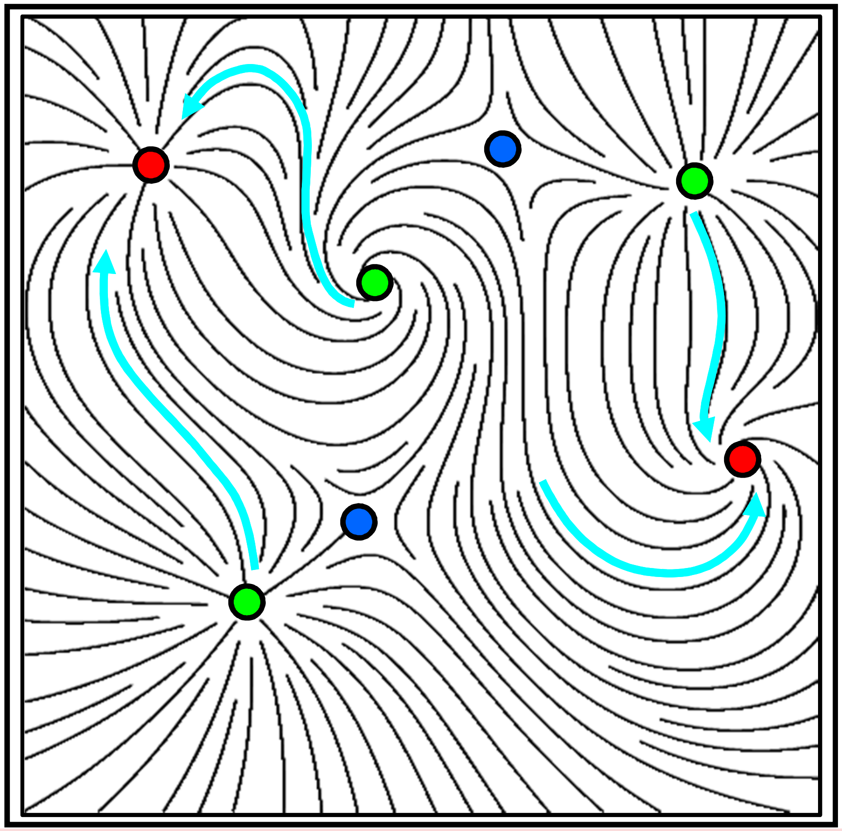
- Limit sets reveal the long-term behaviors of vector fields, correspond to flow recurrence.
- The **limit sets** are:

$$\alpha(\mathbf{x}) = \cap_{t<0} cl(\varphi((-\infty, t), \mathbf{x}))$$

point (or curve) reached after ***backward*** integration by streamline seeded at $\mathbf{x}$

$$\omega(\mathbf{x}) = \cap_{t>0} cl(\varphi((t, \infty), \mathbf{x}))$$

point (or curve) reached after ***forward*** integration by streamline seeded at $\mathbf{x}$



Invariant Sets

- An **invariant set** $S \subset X$ satisfies $\varphi(R,S)=S$
  - A trajectory is an invariant set
  - Fixed points and periodic orbits are *compact* and *disjoint* invariant sets

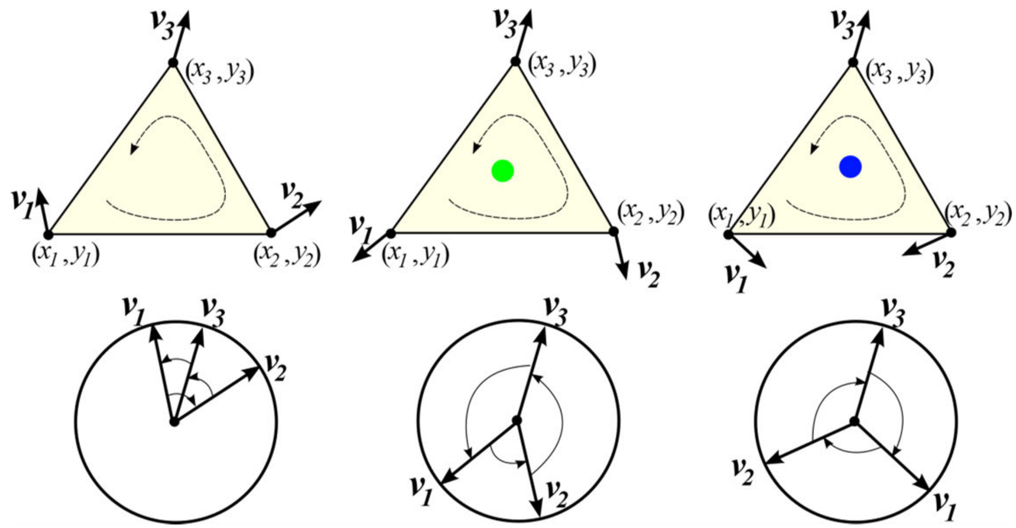
Classifications of Features

Poincaré index I

Sinks, sources, centers: I=1

Saddles: I=-1

Regular, periodic orbits I=0



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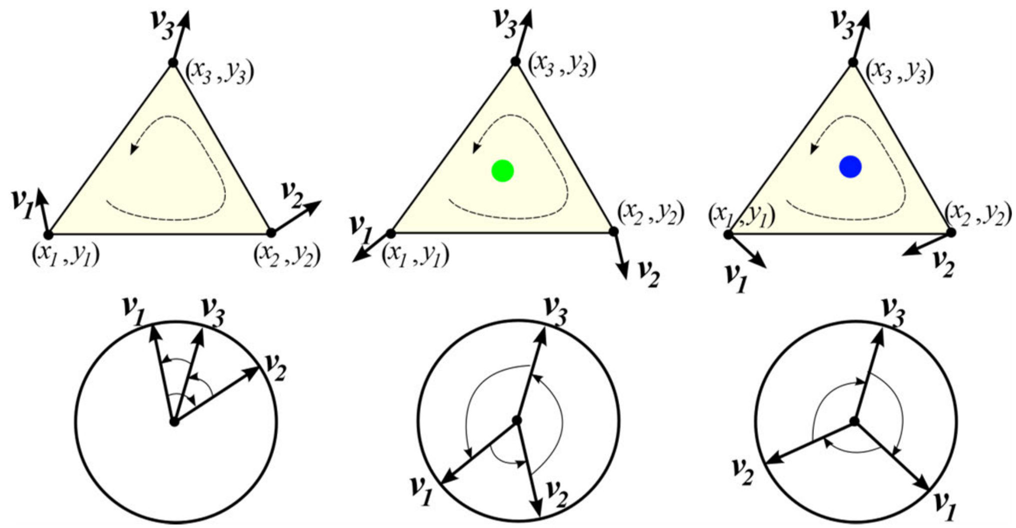
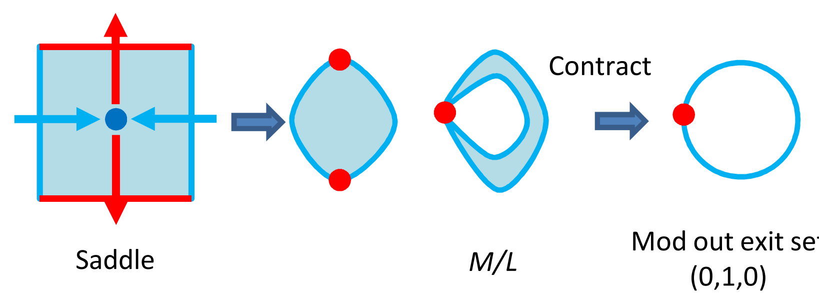


Conley index $C^* = (\beta_0, \beta_1, \beta_2)$ [Conley 78]

Regular flow:	(0,0,0)
An attracting fixed point (e.g. sink):	(1,0,0)
A repelling fixed point (e.g. source):	(0,0,1)
A saddle:	(0,1,0)
An attracting periodic orbit:	(1,1,0)
A repelling periodic orbit:	(0,1,1)



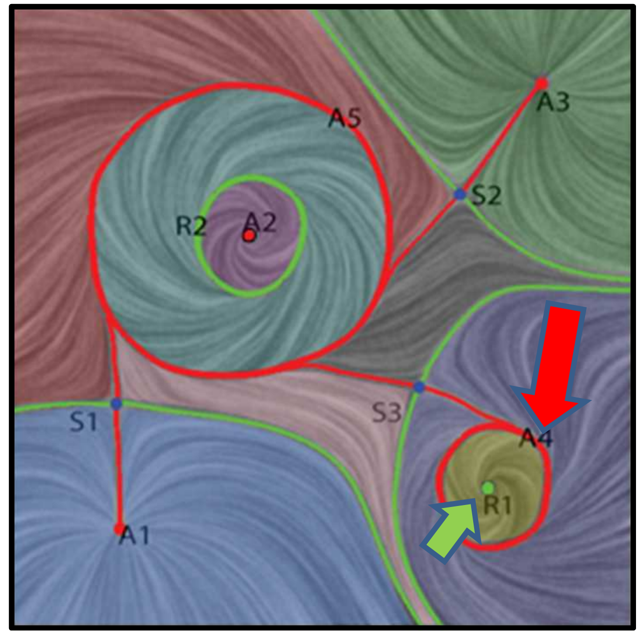
Vector Field Topology – ECG

- An entity connection graph (or ECG) is an extended topological skeleton which consists of [Chen et al. 2007]
  - *Flow recurrent features*

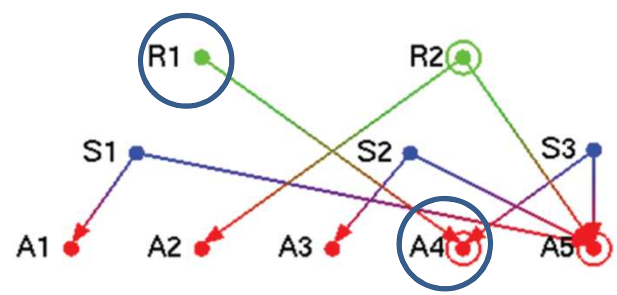
    (fixed points and periodic orbits)
  - *Connectivity*

    (separatrices and others)

It forms a *topological Graph* .

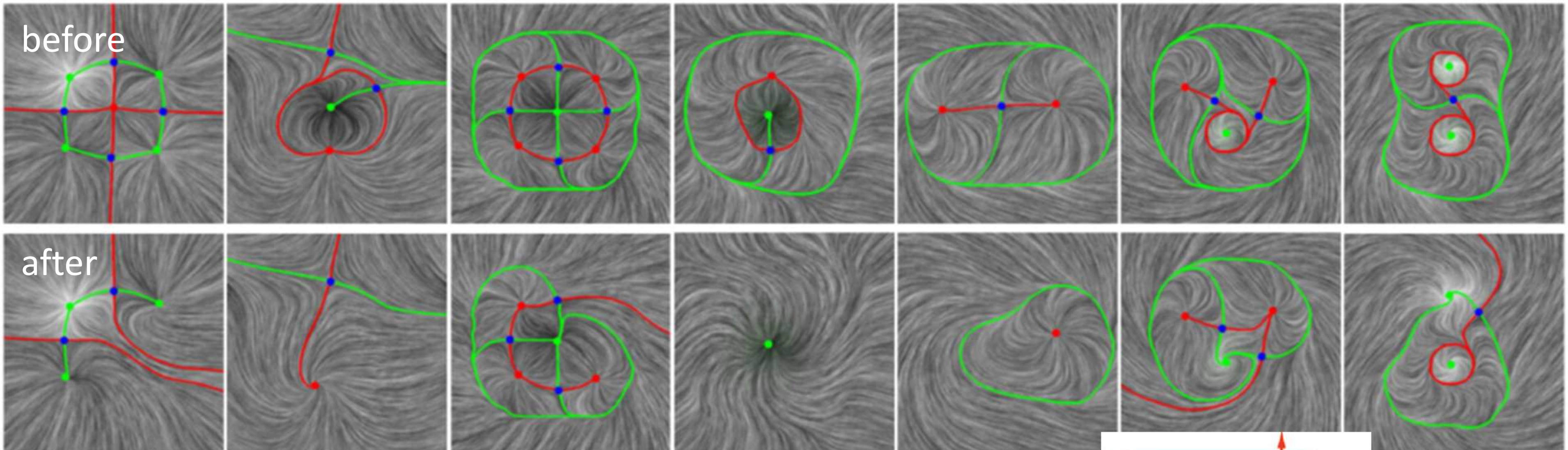


- Three layers based on the **Conley index**
  - Bottom **(A)**ttractors: ($\beta_0 = 1$) sinks, attracting periodic orbits
  - Top **(R)**epellers: ($\beta_2 = 1$) sources, repelling periodic orbits
  - Middle **(S)**addles: ($\beta_1 \neq 0$)

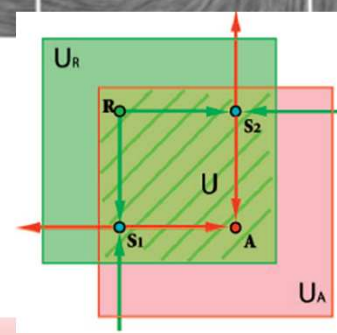


Applications – Simplification

Reduce flow complexity so that people can focus on the more important structure



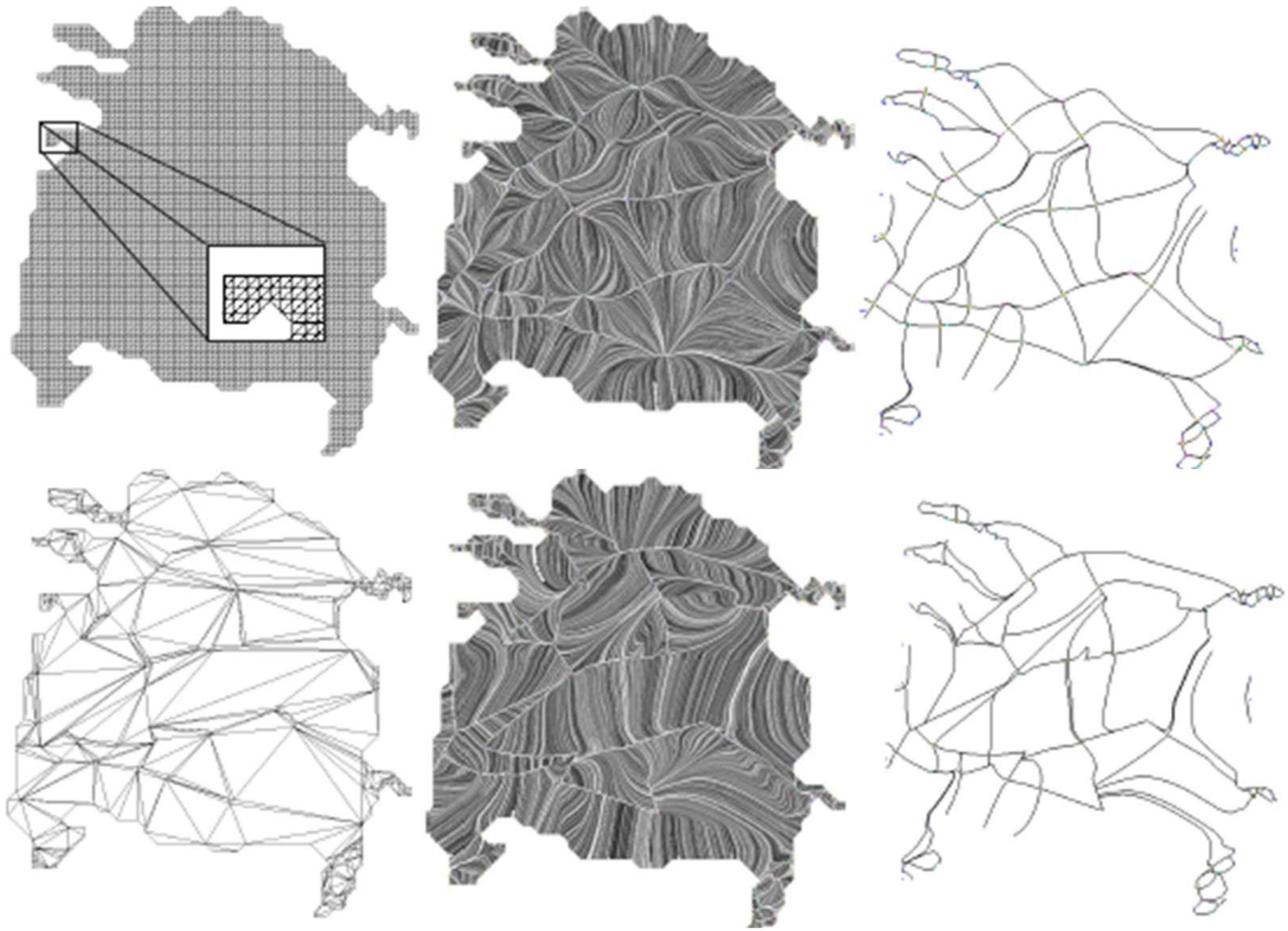
[Chen et al. 2007]



Applications – Data Compression

Before

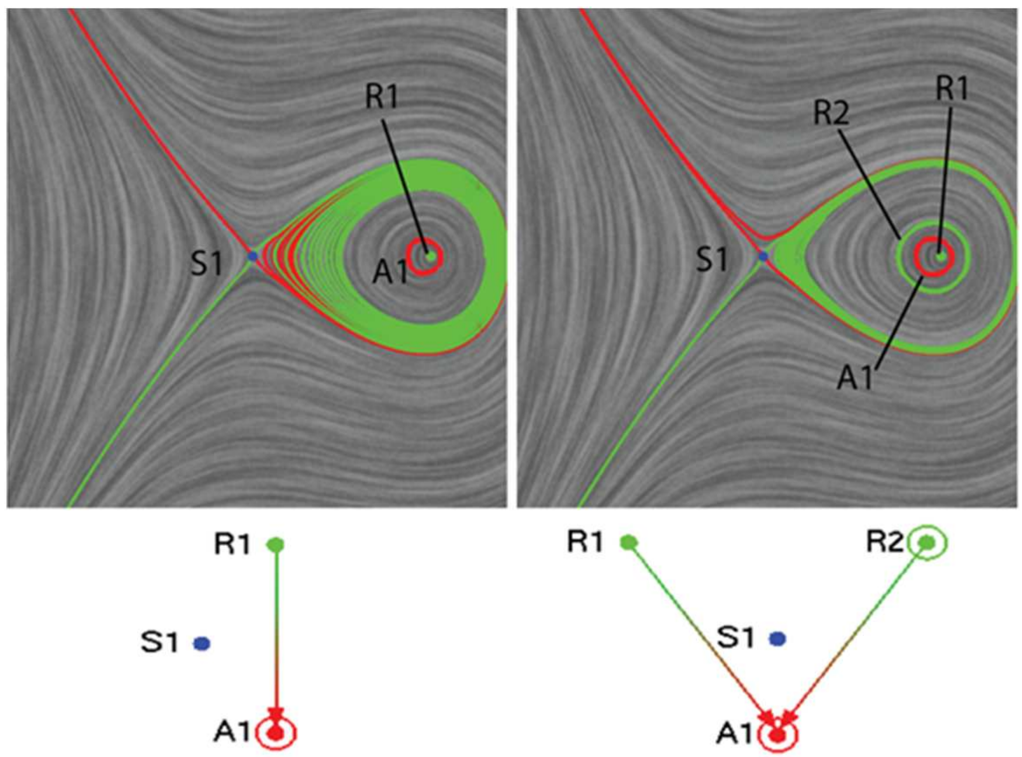
After



[Theisel et al. Eurographics 2003]

Differential Topology is Unstable !

4th Runge Kutta (RK4) | 2nd Runge Kutta (RK2)



ECG

Vector Field Topology

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  - Entity connection graph [Chen et al. TVCG07]
- **Discrete topology**
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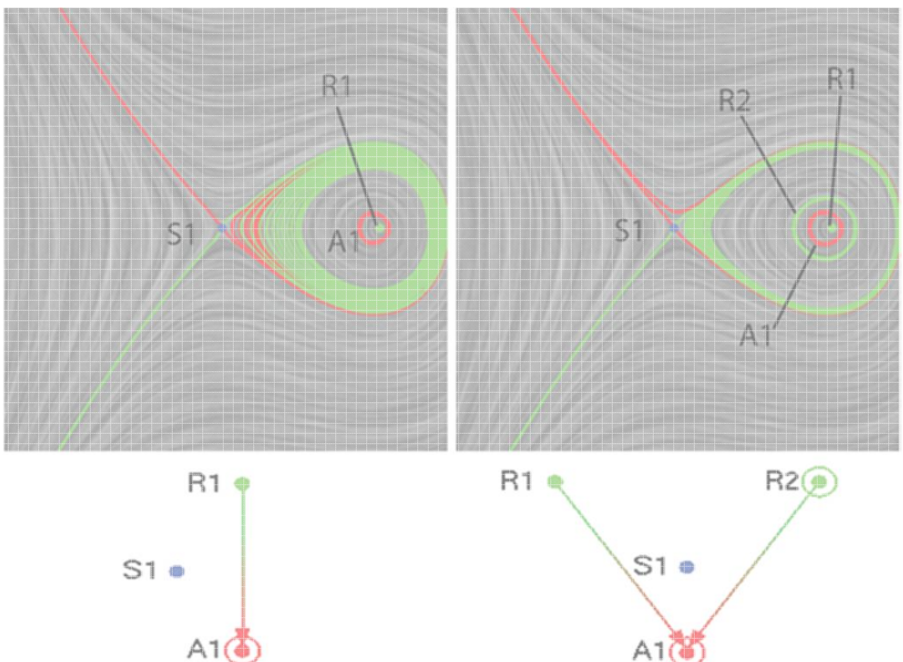
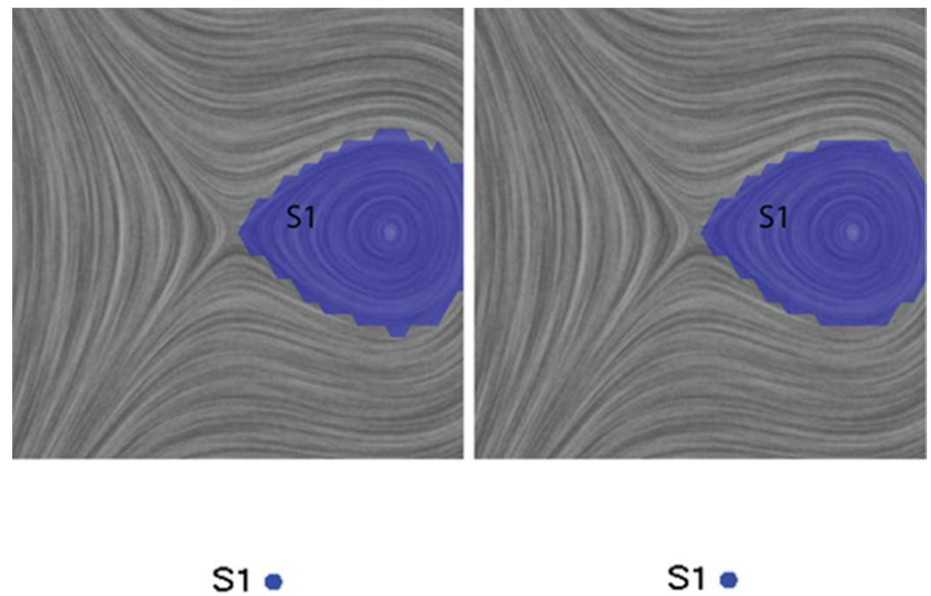
Discrete Topology



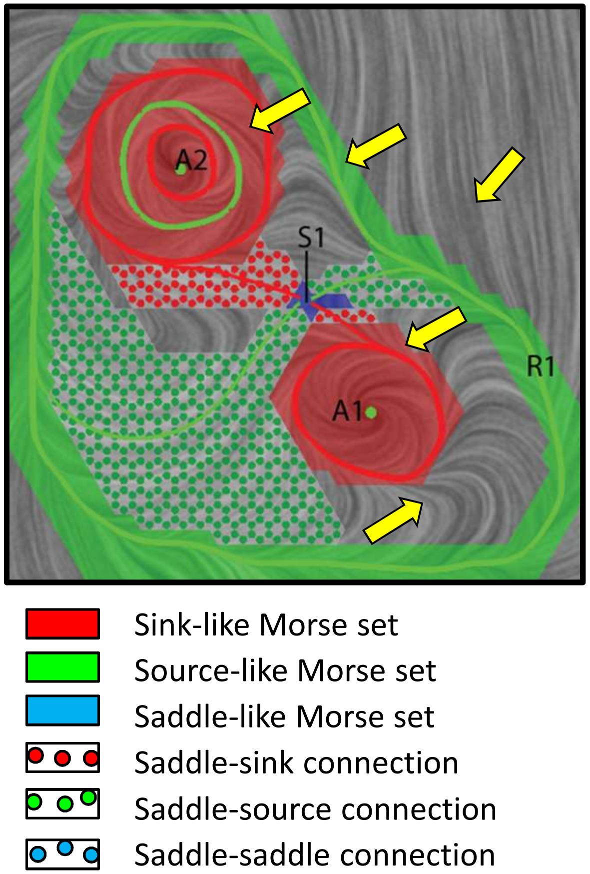
ECG



MCG

Morse Decomposition

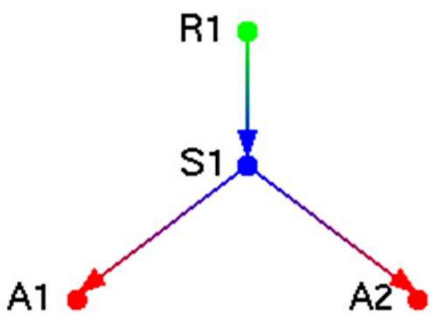
- A ***Morse decomposition*** of surface ***X*** for the flow is a finite collection of disjoint compact invariant sets, called ***Morse sets***.
  - Morse sets capture all flow recurrence (including fixed points and periodic orbits)!
  - Flow outside Morse sets is gradient-like

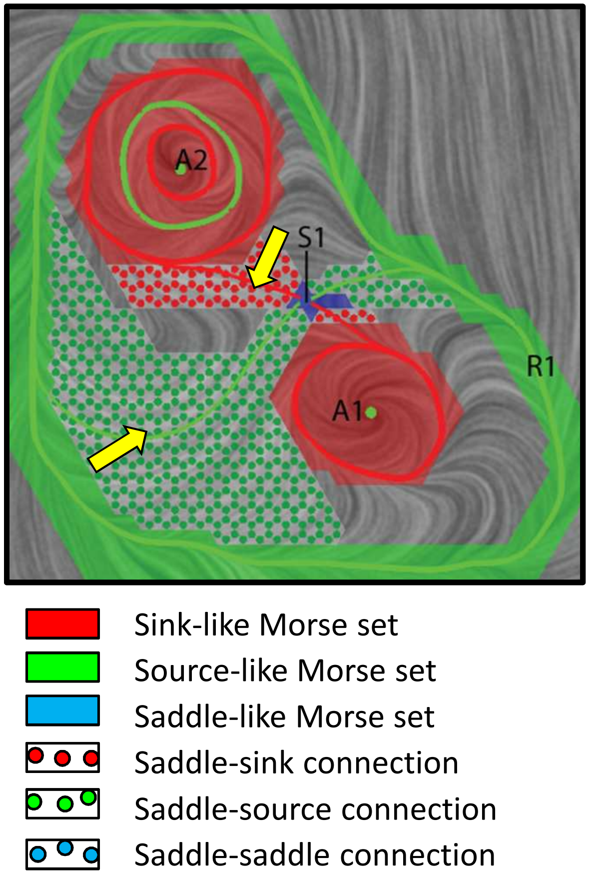


[Chen et al. TVCG08]

Morse Decomposition

- ***Morse connection graph (MCG)***
  - is an acyclic directed graph, whose nodes $\boldsymbol{P}$ are Morse sets, the set of directed edges is a strict partial order $>$

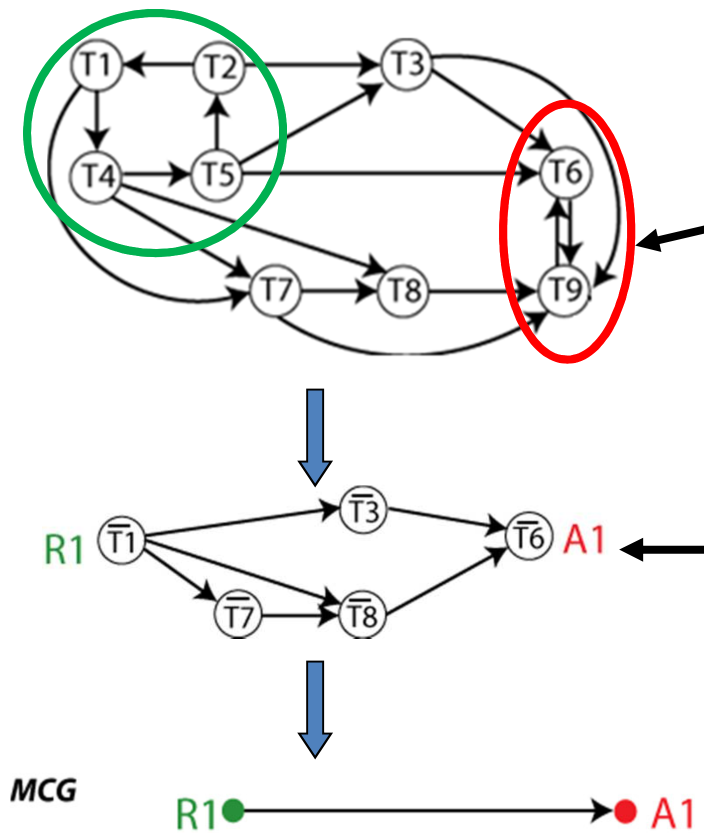




[Chen et al. TVCG08]

The accurate classification of Morse sets is based on Conley index

A Pipeline of Morse Decomposition



Vector field on a triangulation

↓

Flow combinatorialization

↓

Strongly connected component extracting

↓

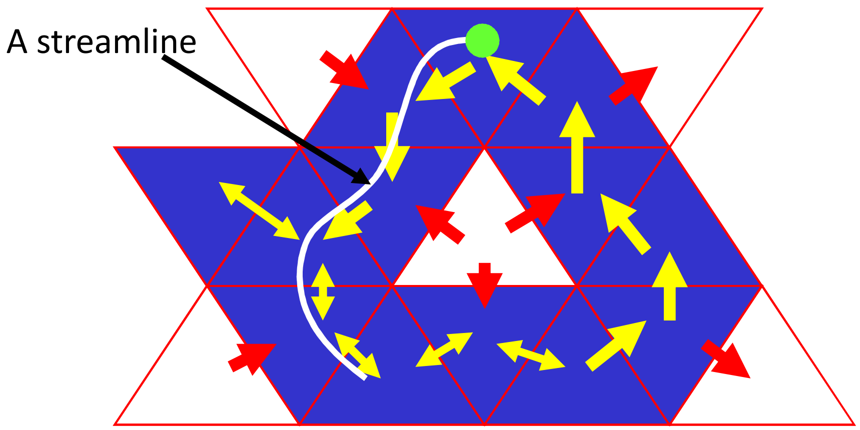
Constructing a quotient graph

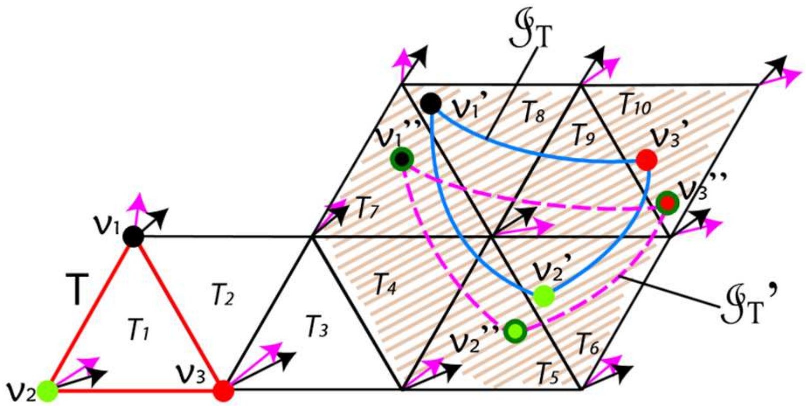
↓

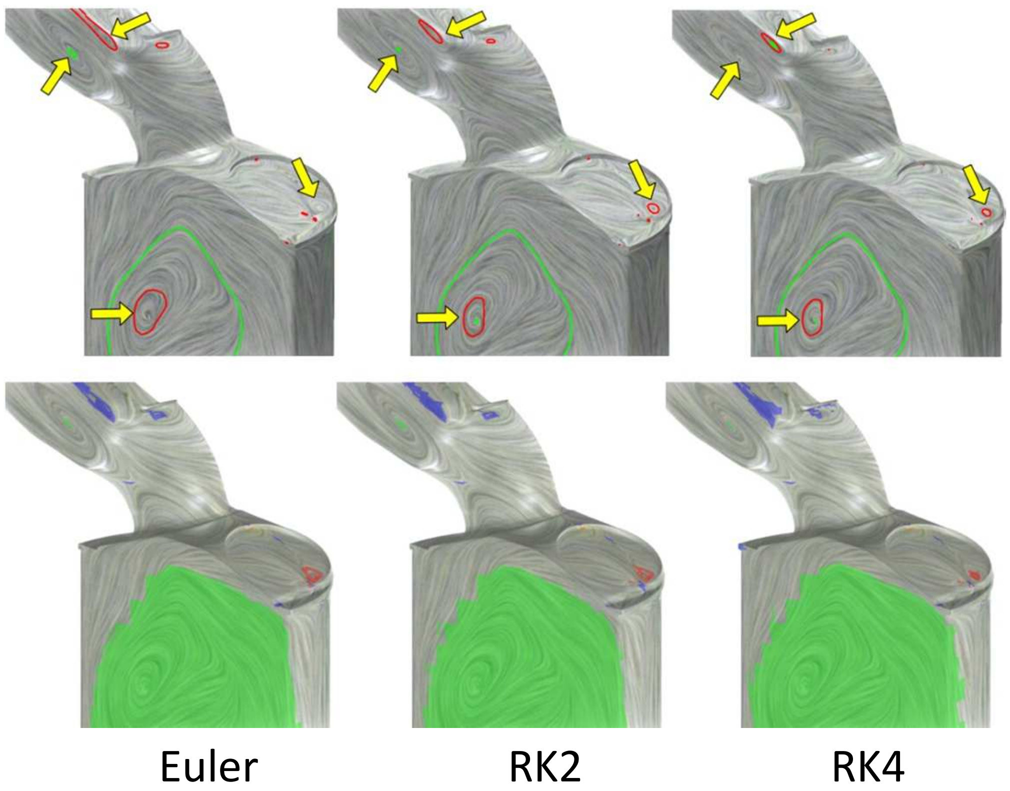
Computing MCG

[Chen et al. TVCG08]

Morse Decomposition is Stable

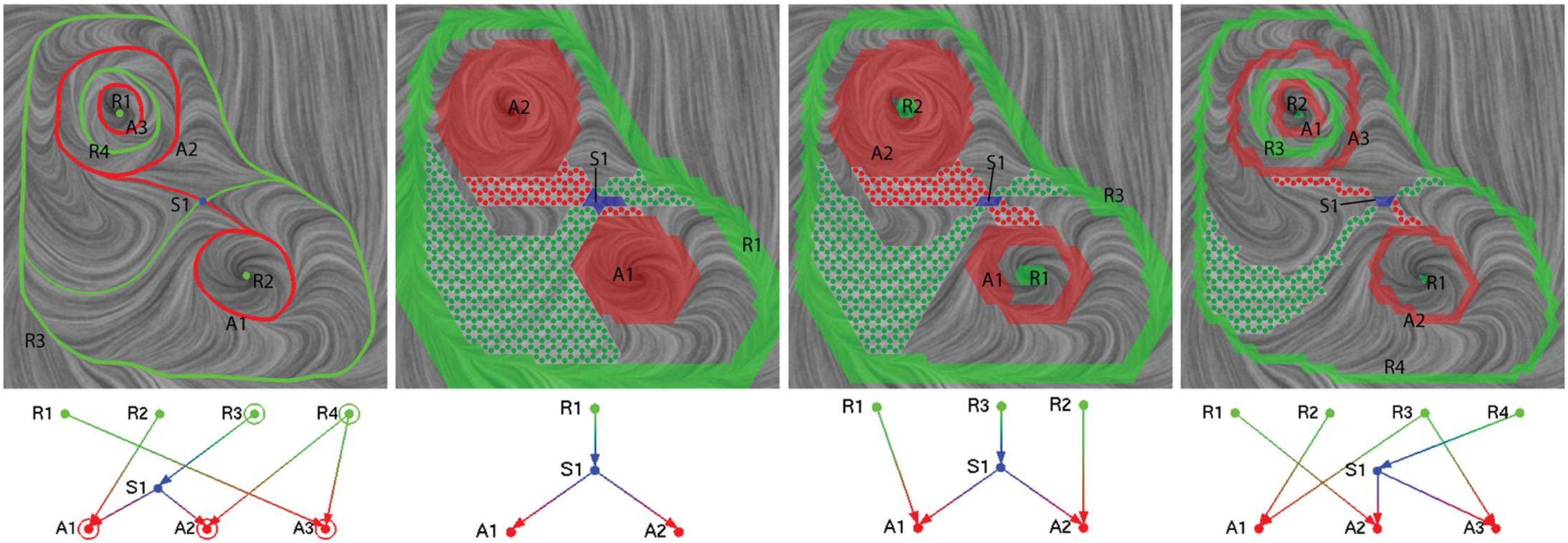






Morse Decomposition is Not Unique

They are all correct!

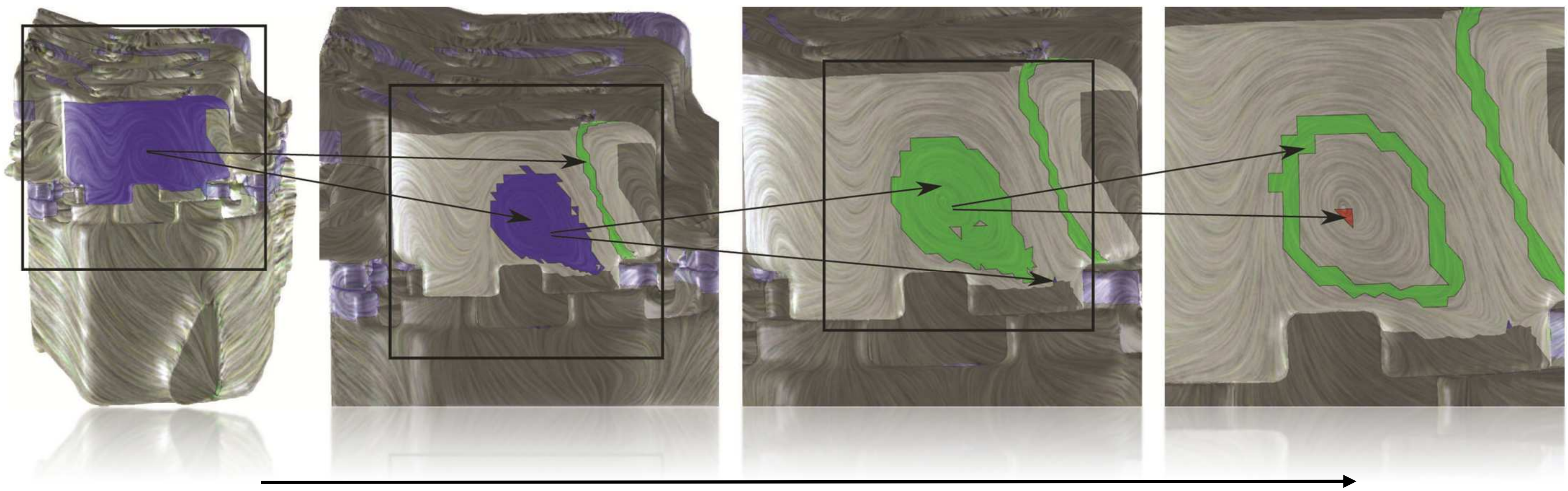


ECG

Small τ → *Large τ*

MCGs with increasing τ

Applications – Hierarchical Structure



Refinement

Automatic vector field simplification

[Chen et al. TVCG12]

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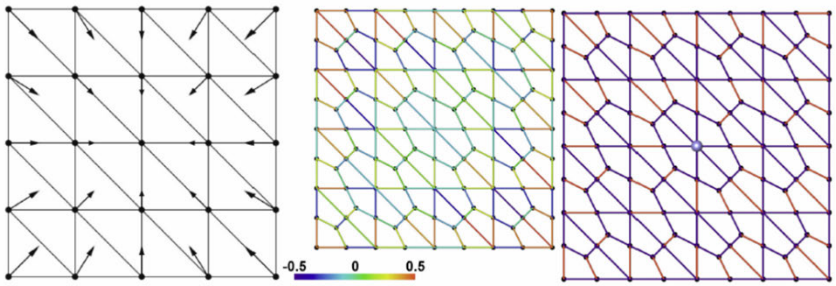

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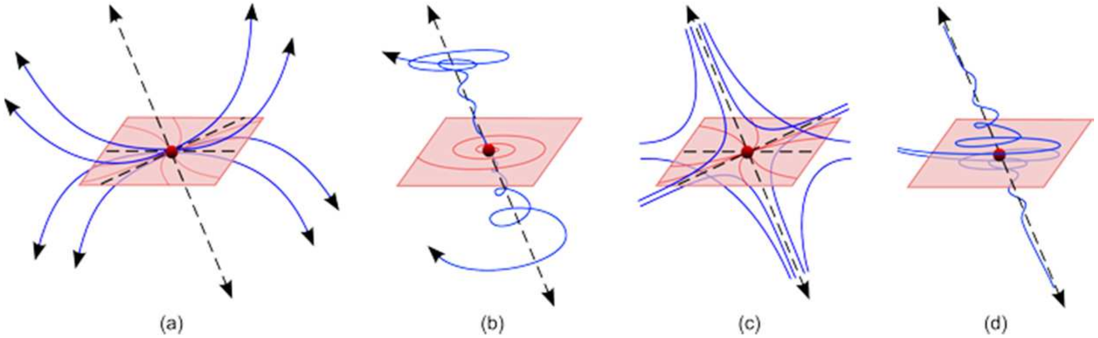
VECTOR FIELD TOPOLOGY IN 3D

3D Flow Topology

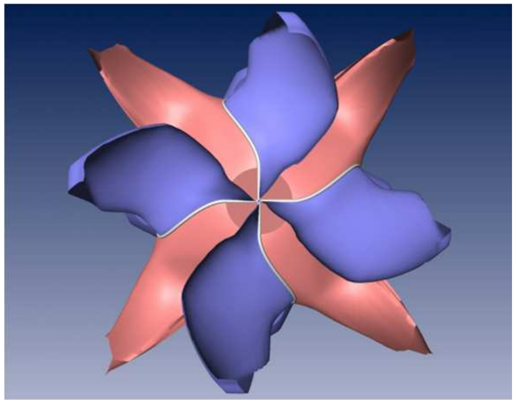
- Similar to 2D case, 3D vector field topology aims to classify the behavior of different streamlines in the domain.
- There are also various flow recurrent dynamics which correspond to those special streamlines, but far more complex than their 2D counterparts.
- 3D flow topology again consists of
  - Fixed points
  - Periodic orbits
  - Their connections including separation structures which can now be both streamline and stream surfaces

3D Flow Topology

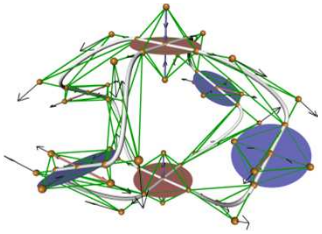
- Fixed points



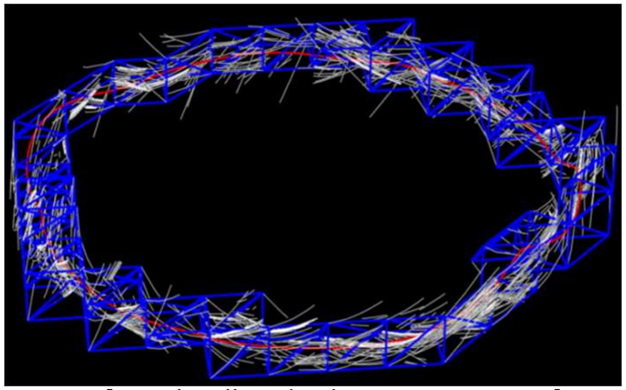
[Peikert and Sadlo http://cgg-journal.com/2010-2/02/index.html]



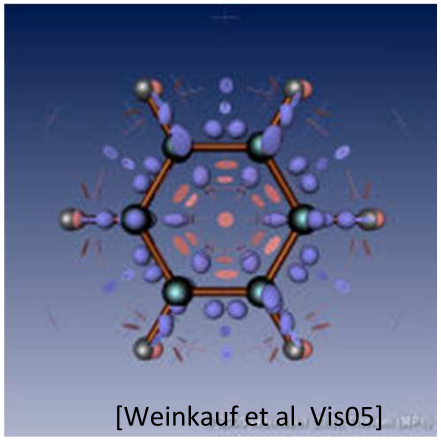
[Weinkauf et al. EG04]



- Periodic orbits

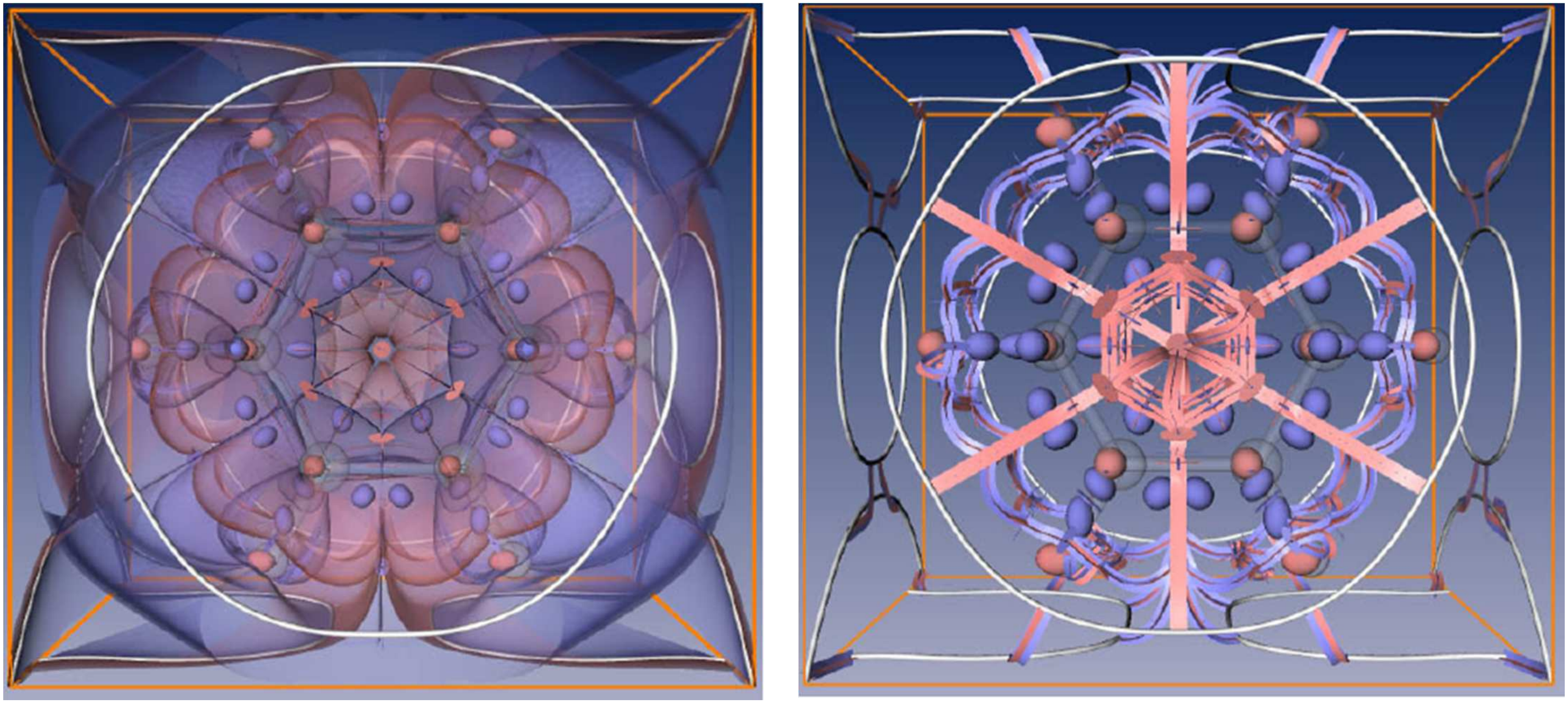


[Wischgoll and Scheuermann 2002]



[Weinkauf et al. Vis05]

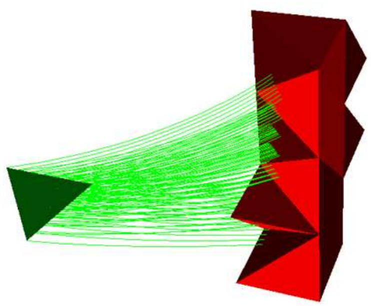
Saddle Connectors

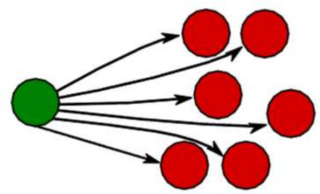


Topological representations of the Benzene data set.
(left) The topological skeleton looks visually cluttered due to the shown separation surfaces.
(right) Visualization of the topological skeleton using saddle connectors.
[Weinkauf et al. VisSym 2004]

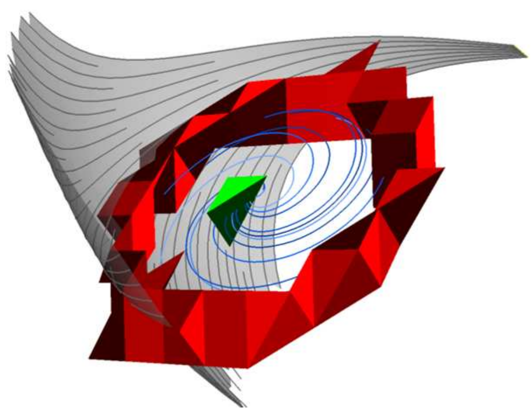
3D Morse Decomposition

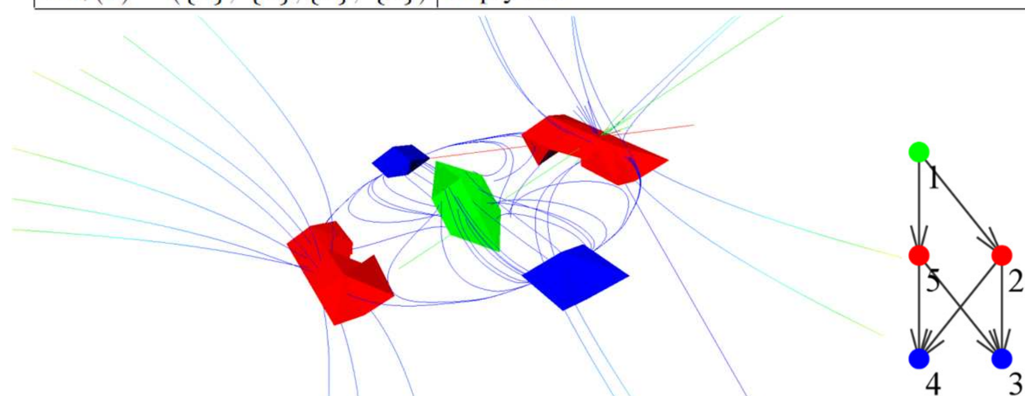
- Similarly, the discrete topology based on Morse decomposition can be directly extended to 3D setting.





Conley index	flow is equivalent to
$CH_*(x) = (\mathbb{Z},\ \{0\},\ \{0\},\ \{0\})$	attracting fixed point
$CH_*(x) = (\{0\},\ \mathbb{Z},\ \{0\},\ \{0\})$	fixed point with one-dimensional unstable manifold
$CH_*(x) = (\{0\},\ \{0\},\ \mathbb{Z},\ \{0\})$	fixed point with two-dimensional unstable manifold
$CH_*(x) = (\{0\},\ \{0\},\ \{0\},\ \mathbb{Z})$	repelling fixed point
$CH_*(\Gamma) = (\mathbb{Z},\ \mathbb{Z},\ \{0\},\ \{0\})$	attracting closed streamline
$CH_*(\Gamma) = (\{0\},\ \mathbb{Z},\ \mathbb{Z},\ \{0\})$	saddle-like closed streamline
$CH_*(\Gamma) = (\{0\},\ \mathbb{Z}_2,\ \mathbb{Z}_2,\ \{0\})$	twisted saddle-like closed streamline
$CH_*(\Gamma) = (\{0\},\ \{0\},\ \mathbb{Z},\ \mathbb{Z})$	repelling closed streamline
$CH_*(\emptyset) = (\{0\},\ \{0\},\{0\},\ \{0\})$	empty set



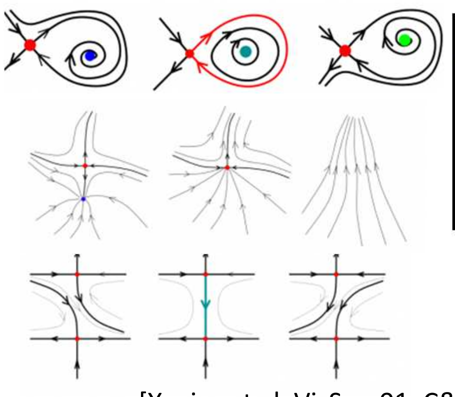


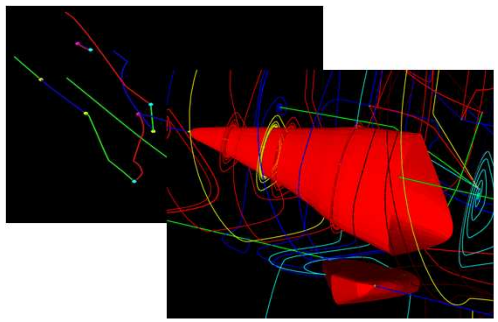
[Reich et al. TopoInVis11]

WHERE ARE WE?

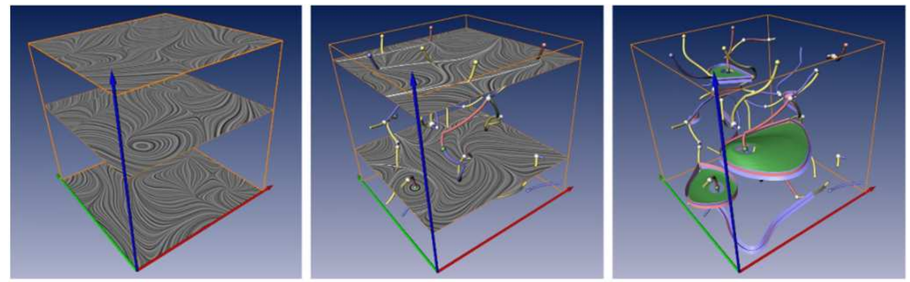
To Time-Dependent Vector Fields

- **Track the Evolution of Instantaneous Topology**





[Xavier et al. VisSym01, C&G02, Vis04]



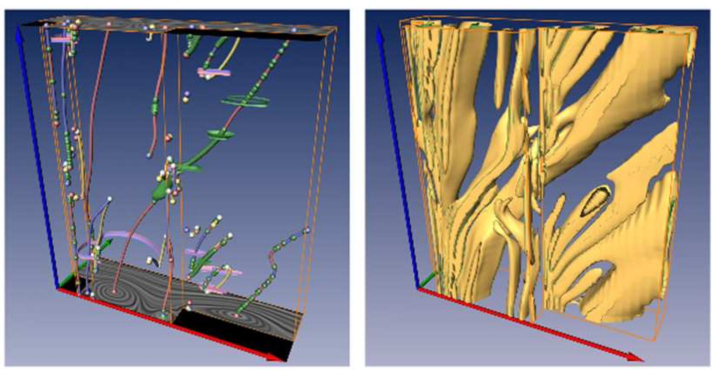
(a) LIC images at 3 different time slices.

(b) Tracking the locations of critical points as stream lines (red/blue/yellow); local bifurcations: Hopf bifurcations (green spheres), fold bifurcations (gray spheres).

(c) Global bifurcations: saddle connections (red/blue flow ribbons), tracked closed stream lines (green surfaces).

[Theisel et al. VisSym2003, Vis04, TVCG05]

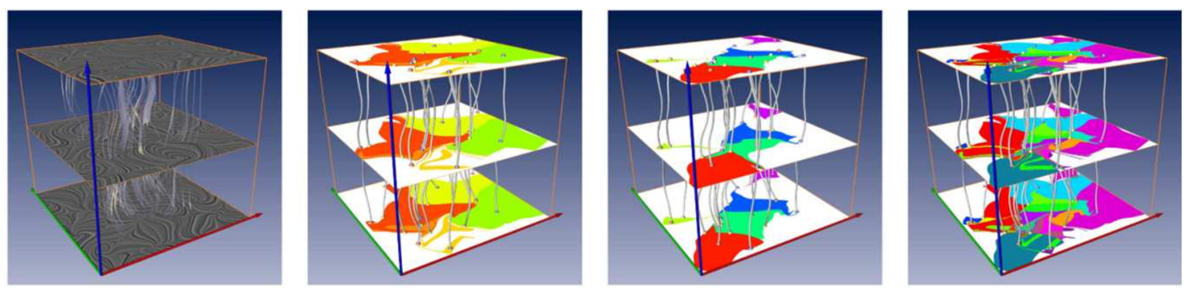
- **Pathline-based**



(b) Stream line oriented topology of the first 100 time steps.

(c) Path line oriented topology of the first 100 time steps.

[Theisel et al. Vis04, TVCG05]



(a) The vector field **p**.

(b) Critical path lines and basins for forward integration.

(c) Critical path lines and basins for backward integration.

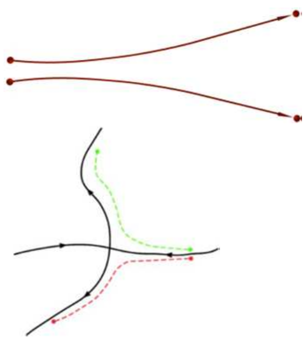
(d) Overlayed basins for forward and backward integration.

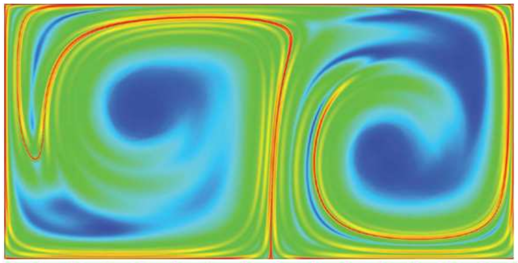
[Shi et al. EuroVis06]

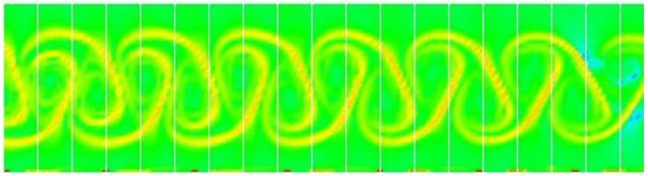
To Time-Dependent Vector Fields

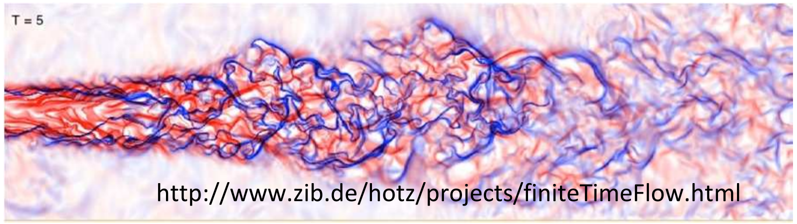
- **FTLE**

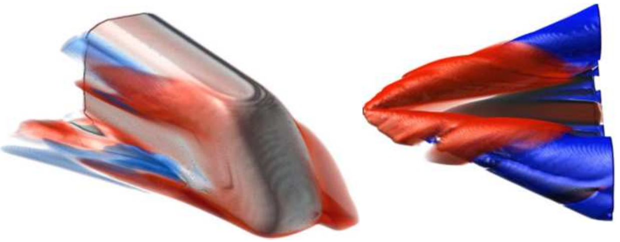
[Haller 2001, Shadden et al. 2005, Garth et al. CGF08, Garth et al. Vis07, Lekien et al. 2007, Sadlo and Peikert TVCG07, Fuchs et al.PG10 etc., Kuhn et al. PacificVis12, etc…]



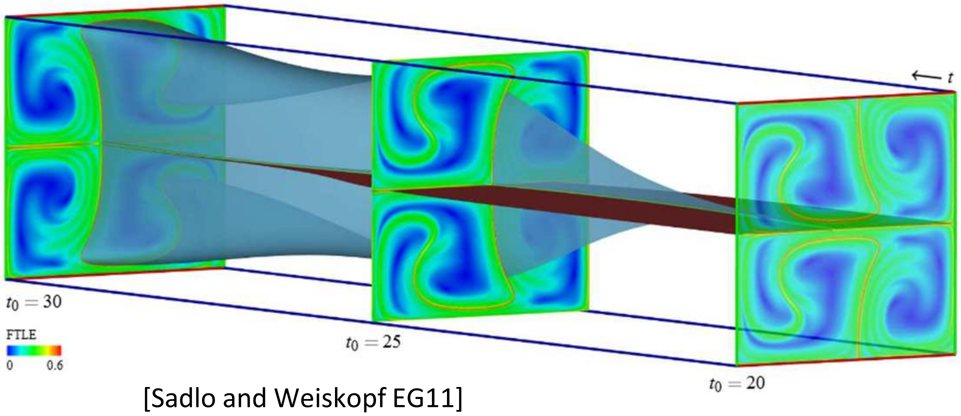




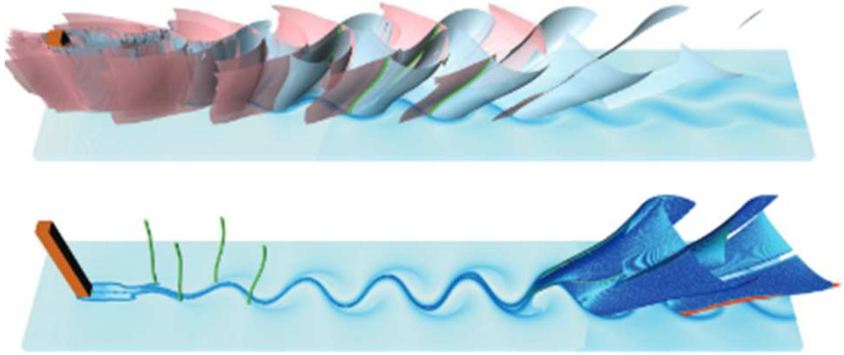




- **Streaklines/Streak-surface based**

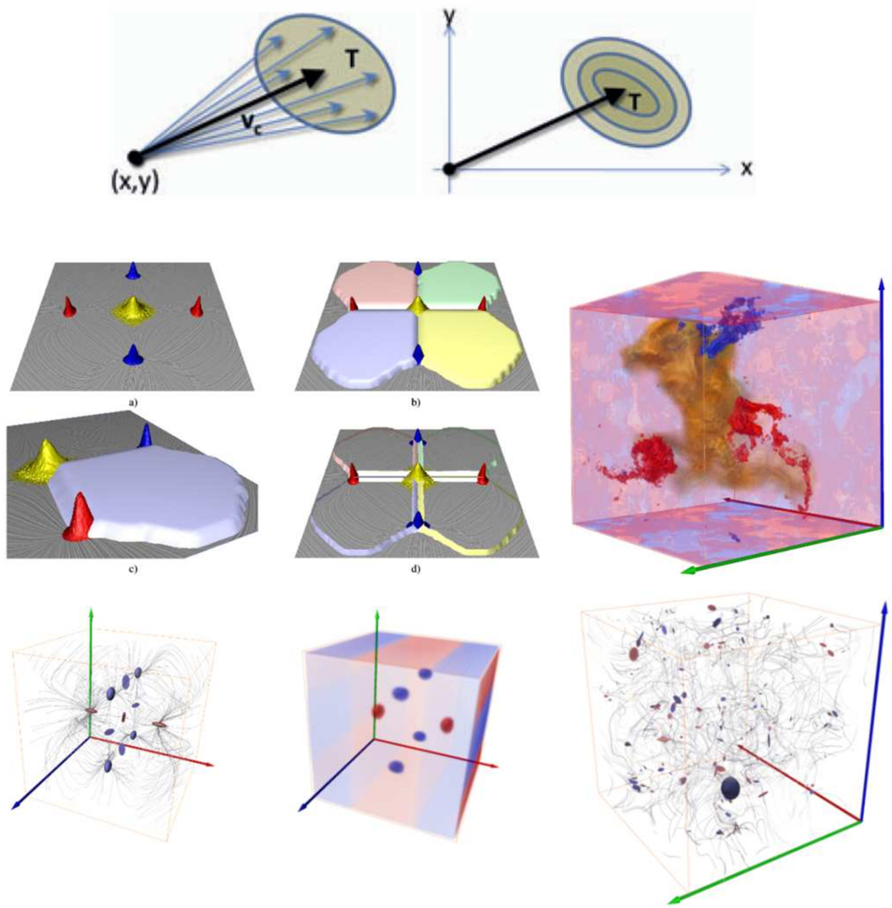


[Sadlo and Weiskopf EG11]

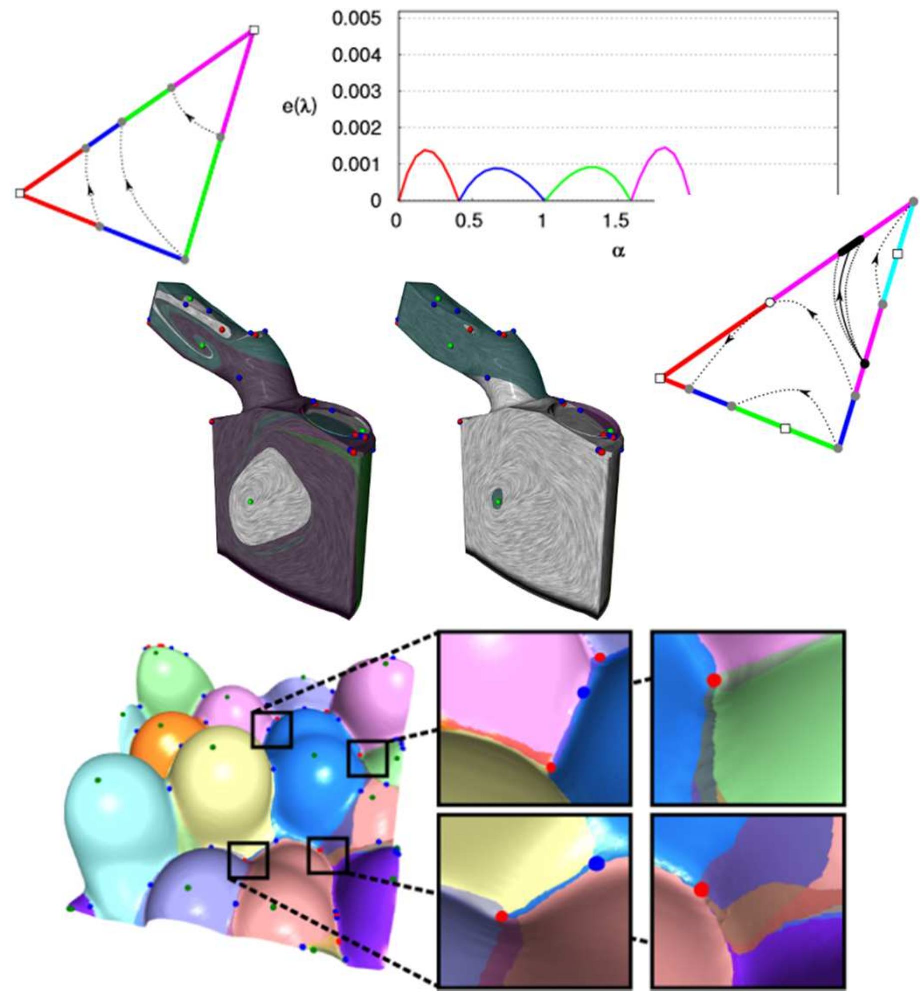


[Uffinger et al. TVCG13]

To Uncertainty Vector Fields



[Otto et al. EG10, PacificVis11]



[Bhatia et al. PacificVis11, TVCG2012]

To Turbulence Flow ?

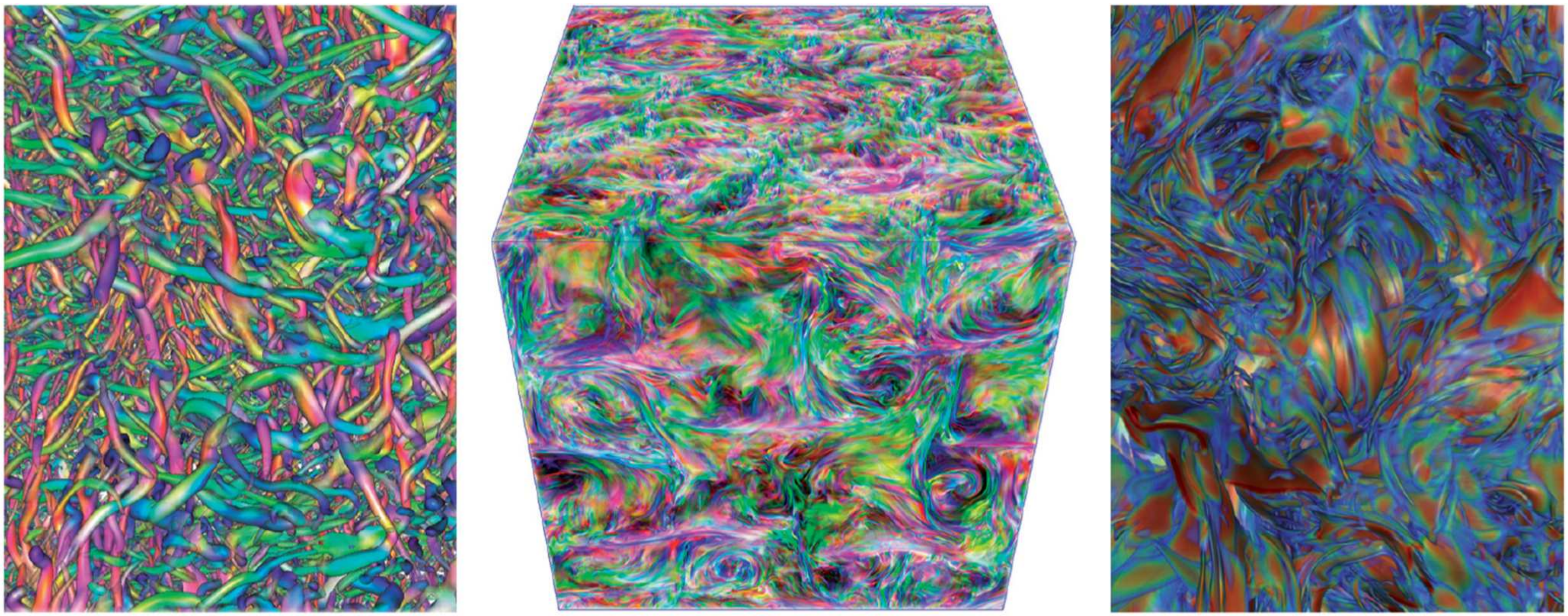


Fig. 1. Visualizations of structures in 1024^3 turbulence data sets on 1024×1024 viewports, directly from the turbulent motion field. Left: Close-up of iso-surfaces of the Δ_{Chong} invariant with direct volume rendering of vorticity direction inside the vortex tubes. Middle: Direct volume rendering of color-coded vorticity direction. Right: Close-up of direct volume rendering of R_S. The visualizations are generated by our system in less than 5 seconds on a desktop PC equipped with 12 GB of main memory and an NVIDIA GeForce GTX 580 graphics card with 1.5 GB of video memory.

[Treib et al. Vis2012]

Thank you and Question?

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