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A Factorial Identity

Either prove the following statement or find a counterexample. For every positive integer a there is a constant C depending only on a such that for every positive integer n we have

$$(an)! \leq C^n (n!)^a$$

Solution This is true. Fix a and let $C = a^a$. Proof is then by induction on n. The base case $n = 1$ is done by observing $a! = a(a-1)(a-2)\cdots 2 \cdot 1 \leq a \cdot a \cdot a \cdots a \cdot a = a^a$. Now assume we have the result for n and we with to show it for $n+1$. Note that

$$\begin{aligned}(a(n+1))! &= (a(n+1))(a(n+1)-1)(a(n+1)-2)\cdots(a(n+1)-a)\cdots 2 \cdot 1 \\ &= (an+a)(an+a-1)(an+a-2)\cdots(an+1)(an)!\end{aligned}$$

From induction we have $(an)! \leq C^n (n!)^a$. Looking at the rest of the right hand side above, we have $(an+a-i) \leq (an+a)$ for all $i < a$. Thus

$$(an+a)(an+a-1)(an+a-2)\cdots(an+1) \leq (an+a)^a$$

Combining the two inequalities then gives

$$(a(n+1))! \leq (an+a)^a C^n (n!)^a$$

Applying an exponential identity and simplifying

$$\begin{aligned}(a(n+1))! &\leq (an+a)^a C^n (n!)^a \\ &= a^a (n+1)^a C^n (n!)^a \\ &= a^a (a^a)^n ((n+1)n!)^a \\ &= C^{n+1} ((n+1)!)^a\end{aligned}$$

as desired.