May 25, 2010

Square Root of a Function

Say a function $g : D \to E$ has a square root $f : D \to E$ if for every x in the domain of g we have $g(x) = f \circ f(x) = f(f(x))$. Can you find a function which does not have a square root? (Hint: there is one with domain and range $[0, 1]$)
Solution: The function $g : [0, 1] \to [0, 1]$ given by

$$g(x) = \begin{cases} 1, & x = 0 \\ x, & 0 < x < 1 \\ 0, & x = 1 \end{cases}$$

does not have a square root.

Proof. Suppose g does have square root f. Then $f \circ f = g$. Now consider the first five numbers in the orbit of 0 under f. Let $a = f(0)$. We will deduce the sequence $0 \mapsto a \mapsto 1 \mapsto a \mapsto \{0, 1\}$ with a contradiction for what the last element should be. By definition $f(a) = f(f(0)) = g(0) = 1$. It is clear $a \notin \{0, 1\}$.[1] Thus $f(1) = g(a) = a$. Finally, $0 = g(1) = f \circ f(1) = f(a) = 1$, which is a contradiction. Thus g does not have a square root. □

[1] if $a = 0$ then $f(0) = 0$, implying $g(0) = 0$. Likewise, if $a = 1$ then $f(0) = 1$ and $g(0) = 1$ giving $f(1) = 1$ and $g(1) = 1$.