April 15, 2007

A Nested Radical

Show the following infinite nested radical is no larger than 2. That is,

$$\sqrt{1+\sqrt{2+\sqrt{3+\sqrt{4+\cdots}}}} < 2$$

You may assume the radical converges.

Solution. Let A represent the value of the nested radical. From algebra we have

$$A^2 - 1 = \sqrt{2+\sqrt{3+\cdots}}$$

and also

$$A\sqrt{2} = \sqrt{2+\sqrt{8+\sqrt{24+\cdots}}}.$$

Comparing the two right hand sides term by term gives the inequality $A^2 - 1 < A\sqrt{2}$, which can be rearranged to be $A^2 - \sqrt{2}A - 1 < 0$. Thinking of the left hand side as the quadratic $x^2 - \sqrt{2}x - 1$, we see that the value of A makes the quadratic negative. The quadratic equation gives the zeros as $\frac{1}{\sqrt{2}}(1 \pm \sqrt{3})$. Since the parabola "points up" it is only negative between its zeros, hence

$$\frac{1}{\sqrt{2}}(1-\sqrt{3}) < A < \frac{1}{\sqrt{2}}(1+\sqrt{3}).$$

But $\frac{1}{\sqrt{2}}(1+\sqrt{3}) < 2$ since $(\frac{1}{\sqrt{2}}(1+\sqrt{3}))^2 = \frac{1}{2}(1+2\sqrt{3}+3) = 2+\sqrt{3} < 4$. Thus $A < 2$. (In fact $\frac{1}{\sqrt{2}}(1+\sqrt{3}) \approx 1.93$.)

Numerical calculation shows $A < 1.7579328$.