February 8, 2007

The Missing Digit of 2^{29}

Consider the following puzzle:

> When evaluated 2^{29} is a nine digit number which is missing exactly one numeral. That is to say it contains every numeral 0–9 except one. Can you determine the missing numeral without direct calculation?

Answer: One needs to consider each digit in the base ten representation of 2^{29}. Let a_0 represent the digit in the ones position, a_1 the digit in the 10's position, and so on up to a_8 to get the nine digit number $a_8a_7a_6a_5a_4a_3a_2a_1a_0$. Write this number algebraically as $a_0 + 10a_1 + 10^2a_2 + \cdots + 10^8a_8$.

Reducing modulo 9 gives $2^{29} = a_0 + 10a_1 + 10^2a_2 + \cdots + 10^8a_8 \equiv a_0 + a_1 + \cdots + a_8 \pmod 9$. Since 2 and 9 are relatively prime Euler's theorem states $2^{\phi(9)} = 2^6 \equiv 1 \pmod 9$ where $\phi(x)$ is the Euler phi function, the number of natural numbers less than x and relatively prime to x. Thus $2^{29} = 2^{4\cdot 6+5} \equiv 2^5 \equiv 5 \pmod 9$. This means the sum of the digits is equal to 5 modulo 9.

Since each digit but one appears exactly once we can calculate the sum of the digits as being between $36 = 0 + 1 + \cdots + 8$ and $45 = 1 + 2 + \cdots + 9$, depending on which digit is missing. There are 10 numbers in this range and only one, 41, is equal to 5 modulo 9. If every digit but 0 were present the sum would be 45. Since $45 - 41 = 4$ the 0 digit must replace the 4 digit. Thus the missing digit is 4.

To check calculate $2^{29} = 536870912$. And the missing digit is 4.