Homework 8: NP-Completeness

CSE 30151 Spring 2017

Due 2017/05/03 at 11:55pm

Instructions

- Create a PDF file (or files) containing your solutions.
- Please name your PDF file(s) as follows:
  - If you're making a complete submission, please name your PDF file `netid-hw8.pdf`, where `netid` is replaced with your NetID.
  - If you're submitting some problems now and want to submit other problems later, name your PDF file `netid-hw8-123.pdf`, where `123` is replaced with the problems you are submitting at this time.
- Submit your PDF file in Sakai. Don't forget to click Submit!

Problems

1. This problem concerns details of the proof of the NP-completeness of *CLIQUE*, which is Theorem 7.32 in the book.
   (a) Convert the formula $\phi = (x \vee x \vee z) \wedge (\bar{x} \vee y \vee y) \wedge (\bar{y} \vee \bar{y} \vee \bar{z})$ into a graph $G = (V, E)$ and integer k, using the construction in the proof of Theorem 7.32 (so that ϕ is satisfiable iff G has a clique of size k).
   (b) Find a truth assignment that satisfies ϕ, convert into a subset of V, and verify that it is a clique in G.
   (c) Convert the formula $\phi = (x \vee x \vee x) \wedge (\bar{x} \vee \bar{x} \vee \bar{x})$ into a graph G and integer k, and verify that ϕ is not satisfiable and G has no clique of size k.
2. In the *knapsack problem*, you are given a knapsack with maximum weight capacity W kilograms, and a collection of k items with weights $w_1, \ldots, w_k$ kilograms. Each item also has a value $v_1, \ldots, v_k$ dollars. The decision version of the problem is: Is there a subset of the items with total weight at most W and total value at least V? Show that this problem is NP-complete.

3. Consider the problem of deciding whether the following puzzle has a solution:

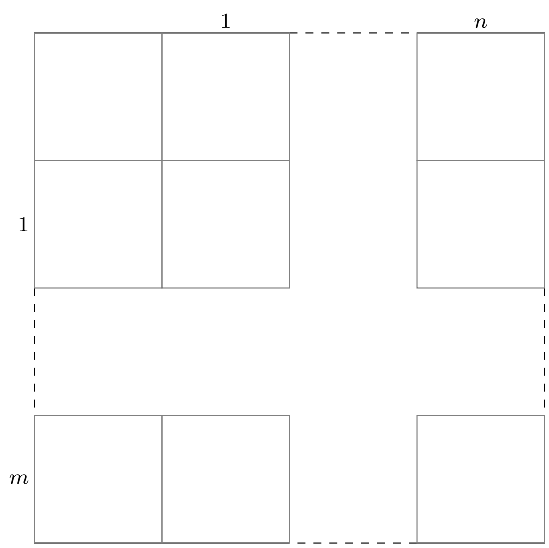
   - You are given a set of square tiles. Each tile has four labels, one on each of its edges. You can make as many copies as you want of each tile. You can't rotate the tiles.
   - You are also given a rectangular frame with labels along the edges.
   - The object is to fill the frame with tiles such that all abutting labels (even blank labels) match.

   In this problem, we will prove that this problem is NP-complete, by reduction from 3SAT. Let $x_1, \ldots, x_n$ be a set of variables, and let ϕ be a formula in 3CNF with m clauses,

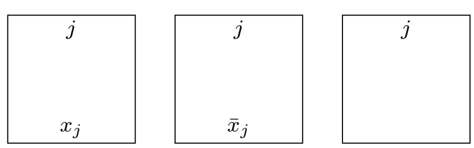
   $$\phi = (a_1 \vee b_1 \vee c_1) \wedge \cdots \wedge (a_m \vee b_m \vee c_m)$$

   where each a_i, b_i, and c_i is a literal, that is, either x_j or $\bar{x}_j$ for some j.

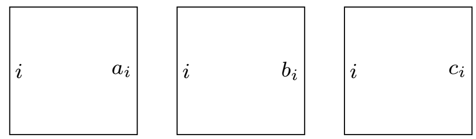
   Define a function f that converts ϕ into an instance of the puzzle as follows. The frame is:



   For each variable x_j, where $j \in \{1, \ldots, n\}$, there are three tiles:

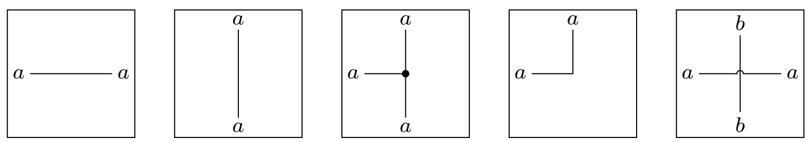


For each clause $(a_i \vee b_i \vee c_i)$, where $i \in \{1, \ldots m\}$, there are three tiles:

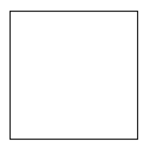


For all literals a and b, that is, a is either x_j or $\bar{x}_j$ for some j, and similarly for b, there are tiles:

Corrected 05/02



Finally, a totally blank tile:



The "wires" drawn in the interior of some tiles are merely suggestive; they don't affect matching at all.

See the Appendix for an example of f applied to a formula. It might be helpful to print and cut the puzzle out and try to solve it.

Prove the following statements:

(a) The set of solvable puzzles is in NP.

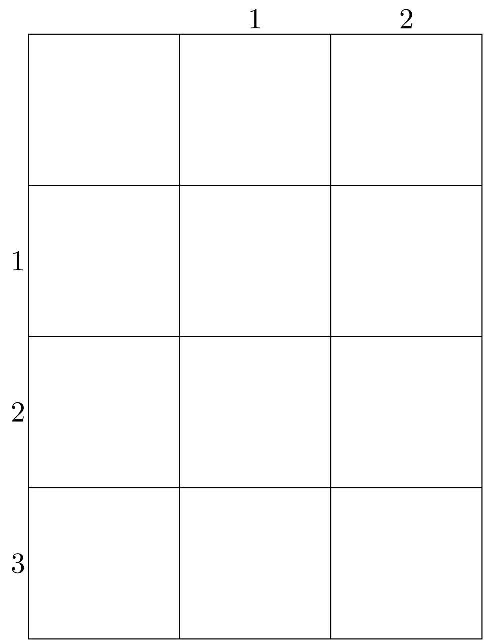
(b) The mapping f is computable in polynomial time.

(c) If ϕ is satisfiable, then the corresponding puzzle $f(\phi)$ is solvable.

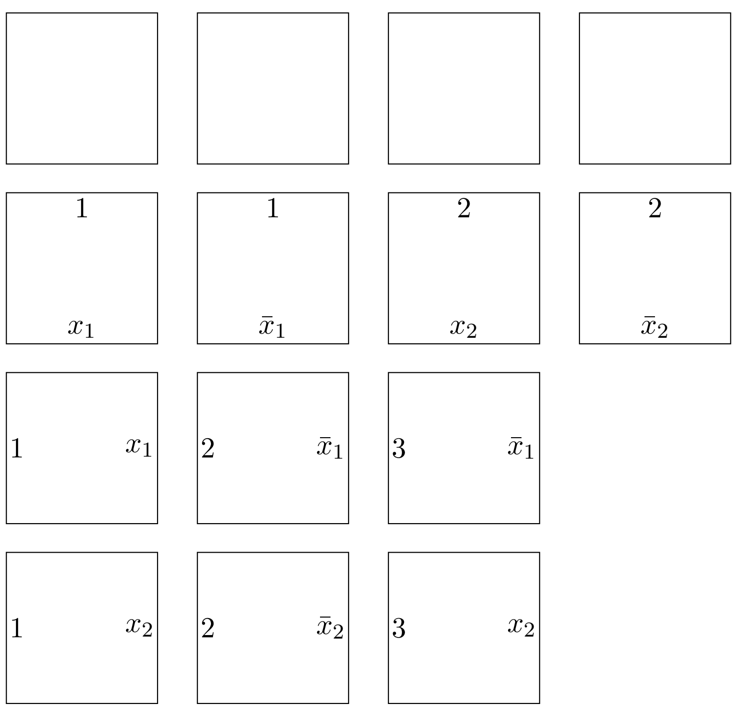
(d) If the puzzle $f(\phi)$ is solvable, then ϕ is satisfiable.

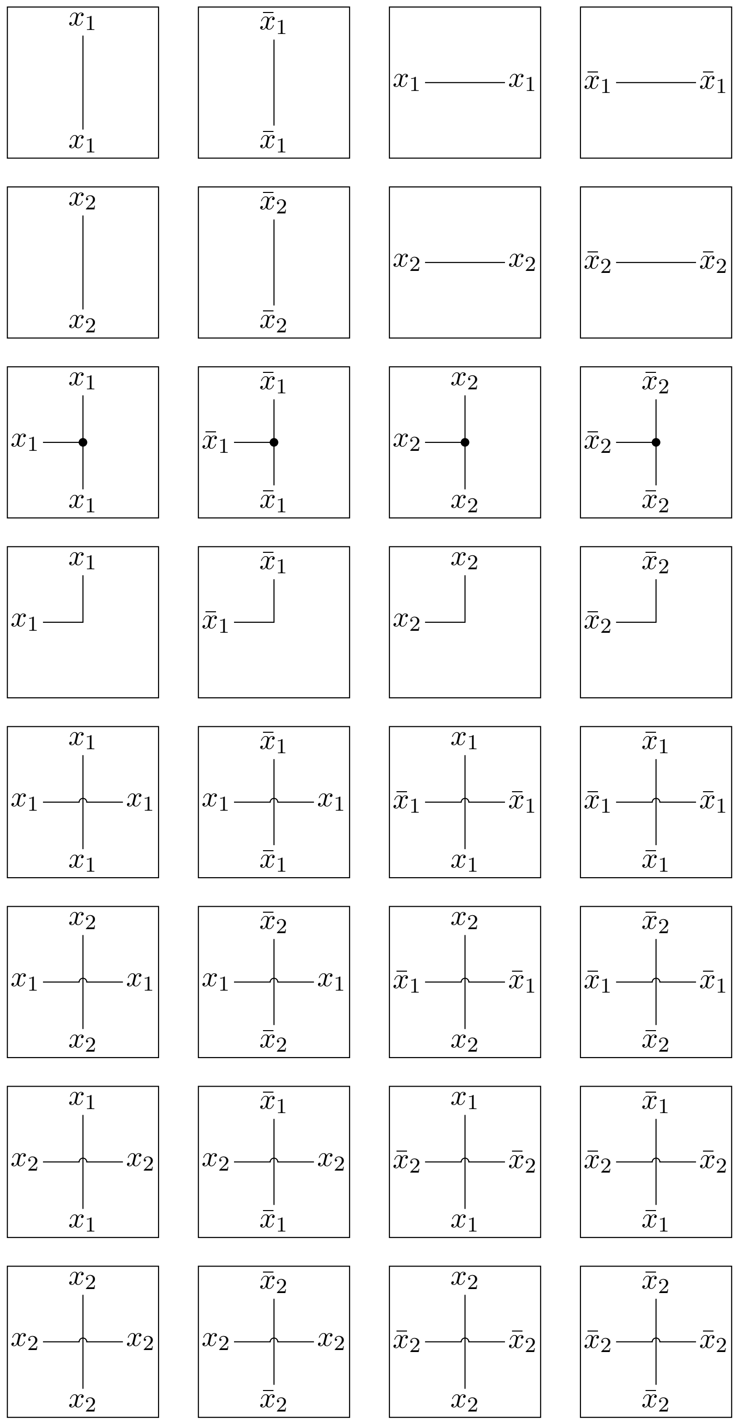
Appendix: Example for Problem 3

If $\phi = (x_1 \vee x_1 \vee x_2) \wedge (\bar{x}_1 \vee \bar{x}_2 \vee \bar{x}_2) \wedge (\bar{x}_1 \vee x_2 \vee x_2)$, the frame would be



And the tiles would be

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