Homework 8: NP-Completeness

Theory of Computing (CSE 30151), Spring 2023

Due: 2023-04-28 11:59pm

Instructions

- Create a PDF file (or files) containing your solutions. You can write your solutions by hand, but please scan them into a PDF.
- Please name your PDF file(s) as follows to ensure that the graders give you credit for all of your work:
  - If you're making a complete submission, name it `netid-hw8.pdf`, where *netid* is replaced with your NetID.
  - If you're submitting some problems now and want to submit other problems later, name it `netid-hw8-123.pdf`, where `123` is replaced with the problem number(s) you are submitting at this time.
- Submit your PDF file(s) in Canvas.

Problems

1. (6 points) This is a warm-up problem on details of the proof of the NP-completeness of *CLIQUE*, which is Theorem 7.32 in the book.

   (a) Convert the formula $\phi = (x \vee x \vee z) \wedge (\bar{x} \vee y \vee y) \wedge (\bar{y} \vee \bar{y} \vee \bar{z})$ into a graph $G = (V, E)$ and integer k, using the construction in the proof of Theorem 7.32 (so that ϕ is satisfiable iff G has a clique of size k).

   (b) For each satisfying truth assignment of ϕ,

   - Please write down the truth assignment.
   - How many subsets of V does it correspond to?
   - Please draw one of them.
   - Is it a clique of G?

   (c) Convert $\phi = (x \vee \bar{y} \vee \bar{y}) \wedge (\bar{x} \vee \bar{x} \vee \bar{x}) \wedge (y \vee y \vee y)$ into a graph G and integer k, and explain why ϕ is not satisfiable and G has no clique of size k.

2. (16 points) In the game of Digits,[1] you are given

   - A target number t (e.g., 56)
   - A multiset S of source numbers (e.g., $\{2, 3, 4, 5, 10, 25\}$)
   - A set $\mathcal{O}$ of operations ($\{+, -, \times, \div\}$)

   and the goal is to write an expression involving the source numbers, operations, and parentheses that evaluates to the target number. You don't have to use every number, but each number can only be used as many times as it occurs in S. If this possible, we say that $(t, s, \mathcal{O})$ is solvable. For example, $(56, \{2, 3, 4, 5, 10, 25\}, \{+, -, \times, \div\})$ is solvable because

   $$2 \times 4 \times (10 - 3) = 56.$$

   Prove that the following languages are NP-complete. Assume that the size of $\langle t, S \rangle$ is the total number of bits in t and S.

   (a) $\{+\}$-DIGITS $= \{\langle t, S \rangle \mid (t, S, \{+\}) \text{ is solvable}\}$ (this one is easy)

   (b) $\{+, \times\}$-DIGITS $= \{\langle t, S \rangle \mid (t, S, \{+, \times\}) \text{ is solvable}\}$

   (c) $\{+, -\}$-DIGITS $= \{\langle t, S \rangle \mid (t, S, \{+, -\}) \text{ is solvable}\}$

   Challenge problem (0 points): Is the full game $\{+, -, \times, \div\}$-DIGITS also NP-complete?

[1] `https://www.nytimes.com/games/digits`

3. (8 points) Define a *neural network* with ℓ inputs and m hidden units to be a function that takes a sequence of inputs

$$x_1, \ldots, x_\ell \in \{0, 1\}$$

and computes

$$h_j = H\left(\sum_{i=1}^{\ell} x_i u_{ij} + a_j\right) \qquad j = 1, \ldots, m$$

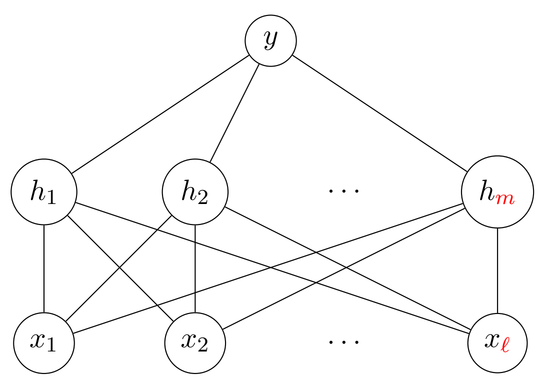
$$y = H\left(\sum_{j=1}^{m} h_j v_j + b\right)$$

where

$$H(z) = \begin{cases} 1 & \text{if } z > 0 \\ 0 & \text{otherwise.} \end{cases}$$

The weights u_{ij}, a_j, v_j, and b can be any rational numbers, and they don't depend on $x_1, \ldots, x_\ell$. The size of the neural network is the number of bits required to store all the weights.

We may visualize the dependencies between the variables like this:



Prove that it is NP-complete to decide, given a neural network with ℓ inputs, whether there is any sequence of inputs $x_1, \ldots, x_\ell$ that makes $y = 1$.