Homework 1: Strings and languages

Theory of Computing (CSE 30151), Spring 2024

Due: 2024-01-26 5pm

Instructions

- Create a PDF file (or files) containing your solutions. You can write your solutions by hand, but please scan them into a PDF.
- Please name your PDF file(s) as follows to ensure that the graders give you credit for all of your work:
  - If you're making a complete submission, name it `netid-hw1.pdf`, where *netid* is replaced with your NetID.
  - If you're submitting some problems now and want to submit other problems later, name it `netid-hw1-part123.pdf`, where `123` is replaced with the problem number(s) you are submitting at this time.
- Submit your PDF file(s) in Canvas under HW1.

Problems (10 points each)

1. **Proof practice.**

   (a) Convert this paragraph proof to a statement–reason proof. Please be sure to write which statement(s) each statement depends on.

   > To show: If s is a string, every substring of a substring of s is a substring of s.
   >
   > Proof: Let y be a substring of s, that is, $s = xyz$ for some x, z; and let v be a substring of y, that is, $y = uvw$ for some u, w. Then $s = xuvwz$, so v is a substring of s.

   (b) Convert this statement–reason proof to a paragraph proof.

   > To show: If w is a string, every prefix of a suffix of w is a suffix of a prefix of w.

1. v is a suffix of w	Given
2. y is a prefix of v	Given
3. $\exists x$ s.t. $xv = w$	(1), def. suffix
4. $\exists z$ s.t. $yz = v$	(2), def. prefix
5. $xyz = w$	(3), (4), substitution
6. xy is a prefix of w	(5), def. prefix
7. y is a suffix of xy	(6), def. suffix

2. **String homomorphisms.** If Σ and Γ are finite alphabets, define a *string homomorphism* to be a function $f\colon \Sigma^* \to \Gamma^*$ that has the property that for any $u, v \in \Sigma^*$,

$$f(uv) = f(u)\, f(v).$$

An example of a string homomorphism is the function that converts hexadecimal numbers into binary numbers, which operates digit-by-digit:

$$\begin{aligned} f_{\text{hb}}\colon \{\texttt{0}, \ldots, \texttt{9}, \texttt{A}, \ldots, \texttt{F}\}^* &\to \{\texttt{0}, \texttt{1}\}^* \\ f_{\text{hb}}(\varepsilon) &= \varepsilon \\ f_{\text{hb}}(\texttt{1}) &= \texttt{0001} \\ f_{\text{hb}}(\texttt{A}) &= \texttt{1010} \\ f_{\text{hb}}(\texttt{1A1A}) &= \texttt{0001101000011010}. \end{aligned}$$

Prove that, in general, every string homomorphism operates by replacing each symbol with a (possibly empty) string. That is, prove that if f is a string homomorphism, then for any $w = w_1 \cdots w_n$ (where $n \geq 0$ and, for $j = 1, \ldots, n$, $w_j \in \Sigma$), we have

$$f(w) = f(w_1) \cdots f(w_n). \qquad (*)$$

Use induction on n.

(a) State and prove the base case ($n = 0$).

(b) Assume that (*) is true for $n = i$ and prove (*) for $n = i + 1$.

You may assume the following facts about strings:

- Identity: For all $x \in \Sigma^*$, $x\varepsilon = x$ and $\varepsilon x = x$.
- Right cancellation: For all $x, y, z \in \Sigma^*$, if $xz = yz$ then $x = y$.
- Left cancellation: For all $x, y, z \in \Sigma^*$, if $xy = xz$ then $y = z$.

3. **Finite and cofinite.** Let $\Sigma = \{\texttt{a}, \texttt{b}\}$. Define $\textsf{FINITE}$ to be the set of all finite languages over Σ, and let $\textsf{coFINITE}$ be the set of all languages over Σ whose *complement* is finite:

$$\textsf{coFINITE} = \{L \subseteq \Sigma^* \mid \overline{L} \in \textsf{FINITE}\}$$

(where $\overline{L} = \Sigma^* \setminus L$). For example, Σ^* is in $\mathsf{coFINITE}$ because its complement is $\emptyset$, which is finite. (Please think carefully about this definition, and note that $\mathsf{coFINITE}$ isn't the same thing as $\overline{\mathsf{FINITE}}$.)

(a) If $L \in \mathsf{FINITE}$, what data structure could you use to represent L, and given a string w, how would you decide whether $w \in L$?

(b) If $L \in \mathsf{coFINITE}$, what data structure could you use to represent L, and given a string w, how would you decide whether $w \in L$?

(c) Are there any languages in $\mathsf{FINITE} \cap \mathsf{coFINITE}$? Prove your answer.

(d) Are there any languages *not* in $\mathsf{FINITE} \cup \mathsf{coFINITE}$? Prove your answer.