

# The Multiplication Principle

**Two step multiplication principle:** Assume that a task can be broken up into two consecutive steps. If step 1 can be performed in  $m$  ways and for each of these, step 2 can be performed in  $n$  ways, then the task itself can be performed in  $m \times n$  ways.

**Example 1** Suppose you have 3 hats, hats A, B and C, and 2 coats, Coats 1 and 2, in your closet. Assuming that you feel comfortable with wearing any hat with any coat. How many different choices of hat/coat combinations do you have? List all combinations.

# The Multiplication Principle

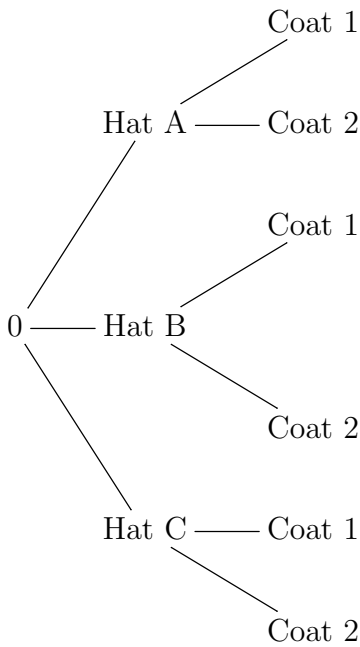
We can get some insight into why the formula holds by representing all options on a tree diagram. We can break the decision making process into two steps here:

Step 1: Choose a hat,

Step 2: choose a coat.

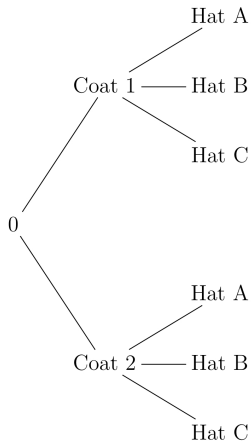
From the starting point 0, we can represent the three choices for step 1 by three branches whose endpoints are labelled by the choice names. From each of these endpoints we draw branches representing the options for step two with endpoints labelled appropriately. The result for the above example is shown below:

# The Multiplication Principle



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Here is the problem done with Coats first and then Hats.



# The Multiplication Principle

Each path on the tree diagram corresponds to a choice of hat and coat. Each of the three branches in step 1 is followed by two branches in step 2, giving us  $3 \times 2$  distinct paths.

If we had  $m$  hats and  $n$  coats, we would get  $m \times n$  paths on our diagram. Of course if the numbers  $m$  and  $n$  are large, it may be difficult to draw.

# The Multiplication Principle

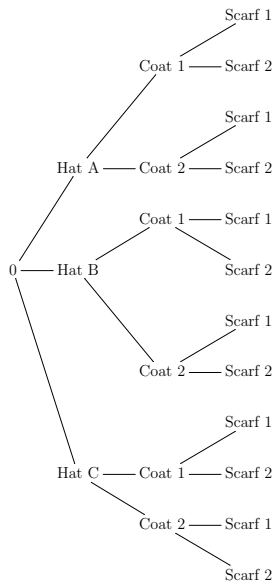
**Example 2** The South Shore line runs from South Bend Airport to Randolph St. Station in Chicago. There are 20 stations at which it stops along the line. How many one way tickets could be printed, showing a point of departure and a destination? (Assuming you can not depart and arrive at the same station.)

You can start at any of twenty stations. Once this is picked, you can pick any of nineteen destinations. The answer is  $20 \cdot 19 = 380$ . If you can get on and off at the same station the answer is  $20 \cdot 20 = 400$ .

# The Multiplication Principle

**Example 3** If your closet contains 3 hats, 2 coats and 2 scarves. Assuming you are comfortable with wearing any combination of hat, coat and scarf, (and you need a hat, coat and scarf today), how many different outfits could you select from your closet? (Break the decision making process into steps and draw a tree diagram representing the possible choices.)

# The Multiplication Principle





# The General Multiplication Principle

If a task can be broken down into  $R$  consecutive steps, Step 1, Step 2, ....., Step  $R$ , and if  
I can perform step 1 in  $m_1$  ways,  
and for each of these I can perform step 2 in  $m_2$  ways,  
and for each of these I can perform step 3 in  $m_3$  ways,  
and so forth  
Then the task can be completed in

$$m_1 \cdot m_2 \cdot \cdots \cdot m_R$$

ways.

Note in example 3,  $R = 3$ ,  $m_1 = 3$ ,  $m_2 = 2$  and  $m_3 = 2$ .

# The General Multiplication Principle

**Example 4** How many License plates, consisting of 2 letters followed by 4 digits are possible?

Would this be enough for all the cars in Indiana?

(Note that it is not a good idea to try to solve this with a tree diagram).

There are 26 letters and 10 digits so the answer is

$$26 \cdot 26 \cdot 10 \cdot 10 \cdot 10 \cdot 10 = 6,760,000$$

The current population of Indiana is around 6,600,000, approximately 5,000,000 over 18 (census.gov); at roughly one car per adult, this would probably be just enough. But in fact Indiana now often uses 3 letters which yields many more possibilities:

$$26 \cdot 26 \cdot 26 \cdot 10 \cdot 10 \cdot 10 = 17,576,000$$

# The General Multiplication Principle

**Example 5** A group of 5 boys and 3 girls is to be photographed.

(a) How many ways can they be arranged in one row?

There are 8 people so there are

$$8 \cdot 7 \cdots 2 \cdot 1 = 8! = 40,320$$

possible ways to do this. The fact that some of them are boys and others girls is irrelevant.

## Example 5 continued

(b) How many ways can the 5 boys and 3 girls be arranged with the girls in front and the boys in the back row?

There are 3 girls so there are  $3 \cdot 2 \cdot 1 = 3!$  ways to arrange the first row. There are 5 boys so there are  $5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5!$  ways to arrange the second row. The two rows can be arranged independently so the answer is  $3! \cdot 5! = 6 \cdot 120 = 720$  possibilities.

# The General Multiplication Principle

**Example 6** How many different 4 letter words (including nonsense words) can you make from the letters of the word  
MATHEMATICS

if (a) letters cannot be repeated (MMMM is not considered a word but MTCS is).

'MATHEMATICS' has 8 distinct letters

{M, A, T, H, E, I, C, S}. Hence the answer is

$$8 \cdot 7 \cdot 6 \cdot 5 = 1,680$$

## Example 6

(b) letters can be repeated (MMMM is considered a word).

There are still only 8 distinct letters so the answer is  $8 \cdot 8 \cdot 8 \cdot 8 = 8^4 = 4,096$ .

(c) Letters cannot be repeated and the word must start with a vowel.





















































The 8 distinct letters {M, A, T, H, E, I, C, S} have 3 vowels {A, E, I}. You can select a vowel in any of 3 ways. Once you have done this you have 7 choices for the second letter; 6 choices for the third letter; and 5 choices for the fourth letter. Hence the answer is  $3 \cdot 7 \cdot 6 \cdot 5 = 630$ .

# The General Multiplication Principle

**A standard deck of 52 cards** can be classified according to suits or denominations as shown in the picture from Wikipedia below. We have 4 suits, Hearts Diamonds, Clubs and Spades and 13 denominations, Aces, Kings, Queens, . . . , twos.

# The General Multiplication Principle

Example set of 52 playing cards; 13 of each suit clubs, diamonds, hearts, and spades

	Ace	2	3	4	5	6	7	8	9	10	Jack	Queen	King
Clubs													
Diamonds													
Hearts													
Spades													



# The General Multiplication Principle

**Example 7** Katy and Peter are playing a card game. The dealer will give each one card and the player will keep the card when it is dealt to them.

(a) How many different outcomes can result?  $52 \cdot 51$

(b) In how many of the possible outcomes do both players have Hearts?  $13 \cdot 12$

## Combining Counting Principles

Recall that the inclusion-exclusion principle says that if  $A$  and  $B$  are sets, then

$$n(A \cup B) = n(A) + n(B) - n(A \cap B) .$$

If the sets  $A$  and  $B$  are **disjoint** then this principle reduces to  $n(A \cup B) = n(A) + n(B)$ . Thus in counting disjoint sets, we can just count the number of elements in each and add. This principle extends easily to  $R > 2$  disjoint sets:

If  $A_1, A_2, \dots, A_R$  are disjoint sets, then

$$n(A_1 \cup A_2 \cup \dots \cup A_R) = n(A_1) + n(A_2) + \dots + n(A_R)$$

## Combining Counting Principles

**Example 8** Katy and Peter are playing a card game. The dealer will give each one card and the player will keep the card when it is dealt to them. In how many of the possible outcomes do both players have cards from the same suit?

There are four distinct possibilities. The possibilities are 2 clubs, 2 diamonds, 2 hearts or 2 spades and these are distinct. In each of these the first card has 13 possibilities while the second has 12. Hence the answer is  $(13 \cdot 12) + (13 \cdot 12) + (13 \cdot 12) + (13 \cdot 12)$ .

A second approach is that there are 52 ways to pick the first card and then there are 12 ways to pick the second. Hence the answer is  $52 \cdot 12$ .

## Combining Counting Principles

**Example 9** Suppose you are going to buy a single carton of milk today. You can either buy it on campus when you are at school, or at the mall when you go to get a gift for a friend or in the neighborhood near your apartment on your way home. There are 5 different shops on campus to buy from, 2 at the mall and 3 in your neighborhood. In how many different shops can you buy the milk?

There are three distinct outcomes. You buy the milk on campus with 5 choices, or you buy the milk at the mall with 2 choices or you buy the milk in your neighborhood with 3 choices, so the answer is  $5 + 2 + 3$ .

If you answered  $5 \cdot 2 \cdot 3$  you answered the question of how many ways could you buy one carton of milk on campus, one carton at the mall and one carton near home. In particular you end up with three cartons.

# Combining Counting Principles

**Example 10** Suppose you wish to photograph 5 schoolchildren on a soccer team. You want to line the children up in a row and Sid insists on standing at the end of the row (either end will do). If this is the only restriction, in how many ways can you line the children up for the photograph? (You can think through this as the number of ways to carry out the task or the number of photographs in a set).

There are two distinct possibilities, Sid is on the left or Sid is on the right. There are  $4!$  ways to arrange the other children. Hence the answer is  $4! + 4!$ .

# Extras, Multiplication Principle

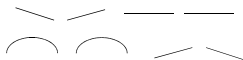
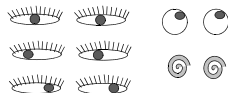
**Example 11** How many faces can you make?

Below you are given 5 pairs of eyes, 4 sets of eyebrows, 2 noses, 5 mouths and 7 hairstyles to choose from. How many possible faces can you make using combinations of the features given if each face you make has a pair of eyes, a pair of eyebrows, a nose, a mouth, and one of the given hairstyles?

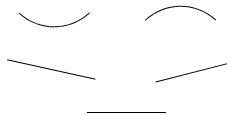
# Example 11 continued - your choices



**Noses**



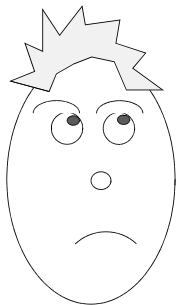
**Eyebrows**



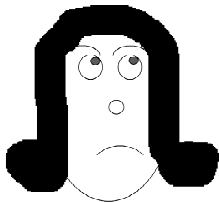
**Mouths**

## Example 11 continued

Here is an example of 3 faces, draw three different faces with the features given!



I don't want a Lisa Simpson Hairdo!



If you say "Multiplication Principle" one more time....



How many roads must a face walk down....

$$5 \cdot 4 \cdot 2 \cdot 5 \cdot 7 = 1,400.$$



# Shakespearean Insults

**Example 12:** How many insults can you make?  
If you follow the directions on the following Shakespeare Insult Kit, how many different insults can you make?

# Shakespeare Insult Kit

To create a Shakespearean insult...

Combine one word from each of the three columns below, prefaced with "Thou":

## Column 1

artless  
bawdy  
beslubbering  
bootless  
churlish  
cockered  
clouted  
craven  
currish  
dankish  
dissembling  
droning  
errant  
fawning  
fobbing  
froward  
frothy  
gleeking  
goatish  
gorbellied  
impertinent  
infectious  
jarring  
loggerheaded  
lumpish  
mammering  
mangled  
mewling  
paunchy  
pribbling  
puking  
puny  
qualling  
rank  
reeky  
roguish  
ruttish  
saucy  
spleeny  
spongy  
surly  
tottering  
unmuzzled  
vain  
venomed  
villainous  
warped  
wayward  
weedy  
yeasty

## Column 2

base-court  
bat-fowling  
beef-witted  
beetle-headed  
boil-brained  
clapper-clawed  
clay-brained  
common-kissing  
crook-pated  
dismal-dreaming  
dizzy-eyed  
doghearted  
dread-bolted  
earth-vexing  
elf-skinned  
fat-kidneyed  
fen-sucked  
flap-mouthed  
fly-bitten  
folly-fallen  
fool-born  
full-gorged  
guts-gripping  
half-faced  
hasty-witted  
hedge-born  
hell-hated  
idle-headed  
ill-breeding  
ill-nurtured  
knotty-pated  
milk-livered  
motley-minded  
onion-eyed  
plume-plucked  
pottle-deep  
pox-marked  
reeling-ripe  
rough-hewn  
rude-growing  
rump-fed  
shard-borne  
sheep-biting  
spur-galled  
swag-bellied  
tardy-gaited  
tickle-brained  
toad-spotted  
unchin-snouted  
weather-bitten

## Column 3

apple-john  
baggage  
barnacle  
bladder  
boar-pig  
bugbear  
bum-bailey  
canker-blossom  
clack-dish  
clotpole  
coxcomb  
dogpiece  
death-token  
dewberry  
flap-dragon  
flax-wench  
flirt-gill  
foot-licker  
fustilarian  
giglet  
gudgeon  
haggard  
harpy  
hedge-pig  
horn-beast  
hugger-mugger  
joithead  
lewdster  
lout  
maggot-pie  
malt-worm  
mammet  
measle  
minnow  
miscreant  
moldwarp  
mumble-news  
nut-hook  
pigeon-egg  
pignut  
puttock  
pumpion  
ratsbane  
scut  
skainsmate  
strumpet  
varlot  
vassal  
whay-face  
wagtail

There are 50 words in each column so the answer is  $50 \cdot 50 \cdot 50 = 125,000$ . (One a day for 342 years)

# Old Exam Questions For Review

**1** Five square tiles of the same size but of different colors (all 5 colors are different) are arranged side by side in a horizontal line. How many different patterns are possible?  
(a)  $2^5$       (b) 5      (c)  $5^2$       (d) 120      (e) 100

The tile in position 1 can be picked 5 ways; the tile in position 2 only 4 ways; and so on. Hence  $5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5! = 120$ . If you want to know how many ways you can lay out a deck of card the answer is  $52!$  (easy to write down) or

80, 658, 175, 170, 943, 878, 571, 660, 636, 856, 403, 766, 975, 289, 505, 440, 883, 277, 824, 000, 000, 000, 000

The factorial notation is extremely useful.

## Old Exam Questions For Review

**2** Pirauellis pizza joint offers a mix and match pizza on its menu. There are 4 different meats to choose from, 5 different vegetables, 4 different types of cheese, and 2 different types of crust. How many different types of Pizza can be made by choosing 1 type of meat, 1 vegetable, 1 cheese and 1 crust?

(a) 80

(b) 4

(c) 20

(d) 160

(e) 49

$$4 \cdot 5 \cdot 4 \cdot 2 = 160.$$