Section 6.4: Permutations

Example Alan, Cassie, Maggie, Seth and Roger want to take a photo in which three of the five friends are lined up in a row. How many different photos are possible?

AMC	AMS	AMR	ACS	ACR
ACM	ASM	ARM	ASC	ARC
CAM	MAS	MAR	CAS	CAR
CMA	MSA	MRA	CSA	CRA
MAC	SAM	RAM	SAC	RCA
MCA	SMA	RMA	SCA	RAC
ASR	MSR	MCR	MCS	CRS
ARS	MRS	MRC	MSC	CSR
SAR	SMR	RMC	CMS	RCS
SRA	SRM	RCM	CSM	RSC
RSA	MRS	CRM	SMC	SCR
RAS	MSR	CMR	SCM	SRC

60, via an exhaustive (and exhausting!) list

Section 6.4: Permutations

Easier, using multiplication principle:

5 options for the person on the left, and

once we've chosen who should stand on the left, 4 options for the position in the middle

and once we've filled both those positions, 3 options for the person on the right.

This gives a total of $5 \times 4 \times 3 = 60$ possibilities.

We have listed all **Permutations** of the five friends taken 3 at a time.

$$\mathbf{P}(5, 3) = 60$$

Permutations

A **permutation** of **n** objects taken **k** at a time is an arrangement of k of the n objects in a specific order. The symbol for this number is $\mathbf{P(n, k)}$.

Remember:

1. A permutation is an arrangement or sequence of selections of objects from a single set.
2. Repetitions are not allowed. Equivalently the same element may not appear more than once in an arrangement. (In the example above, the photo AAA is not possible).
3. the order in which the elements are selected or arranged is significant. (In the above example, the photographs AMC and CAM are different).

Permutations

Example Calculate $\mathbf{P}(10,3)$, the number of photographs of 10 friends taken 3 at a time.

$\mathbf{P}(10,3) = 10 \cdot 9 \cdot 8 = 720.$
Note that you start with 10 and multiply 3 numbers.

A general formula, using the multiplication principle:

$$\boxed{\mathbf{P}(n,k) = n \cdot (n-1) \cdot (n-2) \cdots (n-k+1).}$$

Note that there are k consecutive numbers on the right hand side.

Permutations

Example In how many ways can you choose a President, secretary and treasurer for a club from 12 candidates, if each candidate is eligible for each position, but no candidate can hold 2 positions? Why are conditions 1, 2 and 3 satisfied here?

$\mathbf{P}(12, 3) = 12 \times 11 \times 10 = 1,320.$
Condition 1 is satisfied because we have a single set of 12 candidates for all 3 positions.
Condition 2 is satisfied because no one can hold more than one position.
Condition 3 is satisfied because being president is different than being treasurer or secretary.

Permutations

Example You have been asked to judge an art contest with 15 entries. In how many ways can you assign 1^{st} , 2^{nd} and 3^{rd} place? (Express your answer as $\mathbf{P}(n, k)$ for some n and k and evaluate.)

$\mathbf{P}(15, 3) = 15 \cdot 14 \cdot 13 = 2,730.$

Example Ten students are to be chosen from a class of 30 and lined up for a photograph. How many such photographs can be taken? (Express your answer as $\mathbf{P}(n, k)$ for some n and k and evaluate.)

$\mathbf{P}(30, 10) = 30 \cdot 29 \cdot 28 \cdot 27 \cdot 26 \cdot 25 \cdot 24 \cdot 23 \cdot 22 \cdot 21.$ Note $30 - 10 = 20$ and we stopped at 21.

$\mathbf{P}(30, 10) = 109,027,350,432,000$

Factorials

Example In how many ways can you arrange 5 math books on a shelf?

$\mathbf{P}(5,5) = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$

The number $\mathbf{P}(n,n) = n \cdot (n-1) \cdot (n-2) \cdots 1$ is denoted by $n!$ or "n factorial".

$n!$ counts the number of ways that n objects can be arranged in a row.

$n!$ grows *fast*: $1! = 1$, $2! = 2$, $2! = 6$, $4! = 24$, $5! = 120$, $6! = 720$, $7! = 5,040$, $8! = 40,320$, $9! = 362,880$, $10! = 3,628,800$, … $59! \approx 10^{80}$ (roughly the number of particles in the universe)

Factorials

We can rewrite our formula for $\mathbf{P}(n, k)$ in terms of factorials:

$$\boxed{\mathbf{P}(n, k) = \frac{n!}{(n-k)!}.}$$

Example (a) Evaluate 12!

(b) Evaluate $\mathbf{P}(12, 5)$.

$12! = \mathbf{P}(12, 12) = 12 \cdot 11 \cdots 2 \cdot 1 = 479,001,600.$

$\mathbf{P}(12, 5) = \frac{12!}{7!} = \frac{479{,}001{,}600}{5{,}040} = 95,040.$

Factorials

Example In how many ways can 10 people be lined up for a photograph?

$10! = \mathbf{P}(10, 10)$.

Example How many three letter words (including nonsense words) can you make from the letters of the English alphabet, if letters cannot be repeated? (Express your answer as $\mathbf{P}(n, k)$ for some n and k and evaluate.)

$\mathbf{P}(26, 3) = 15,600$.

Permutations of objects with some alike

Example How many words can we make by rearranging the letters of the word BEER?

The set $\{B, E, E, R\} = \{B, E, R\}$ but we really have 4 letters with which to work. So let us start with the set $\{B, R, E\ , E\ \}$. We arrange them in $4! = 24$ ways:

B R E E	B E R E	B E E R	R B E E	R E B E	R E E B	E B R E	E B E R	E E B R	E R B E	E R E B	E E R B
B R E E	B E R E	B E E R	R B E E	R E B E	R E E B	E B R E	E B E R	E E B R	E R B E	E R E B	E E R B

If we can't tell the difference between E and E (they are both just E), then the words group into *pairs*, e.g., $EEBR$ and $EEBR$ group together — both are the word EEBR.

Thus the number of different words we can form by rearranging the letters must be

$$4!/2 = \frac{4!}{2!}$$

Note that 2! counts the number of ways we can permute the two E's in any given arrangement.

Permutations of objects with some alike

In general the number of permutations of n objects with r of the objects identical is

$$\boxed{\frac{n!}{r!}}$$

Note that $\dfrac{n!}{r!} = \mathbf{P}(n, n-r)$.

We can see this as follows. We have n positions to fill. $\mathbf{P}(n, n-r)$ is the number of ways to put the $n-r$ elements which are unique into the n positions. Once we have done this we just fill in the remaining positions with the repeated element.

From our previous example:

B R _ _	B _ R _	B _ _ R	R B _ _	R _ B _	R _ _ B	_ B R _	_ B _ R	_ _ B R	_ R B _	_ R _ B	_ _ R B

Permutations of objects with some alike

Example How many words (including nonsense words) can be made from rearrangements of the word ALPACA?

$\frac{6!}{3!} = \frac{720}{6} = 120$. There are 6 letters in ALPACA and one of them, 'A' is repeated 3 times.

Example How many words can be made from rearrangements of the word BANANA?

$\{B, A, N, A, N, A\} = \{A, B, N\}$.
The 'A' is repeated 3 times.
The 'N' is repeated 2 times.
The 'B' is repeated once.
Hence the answer is $\frac{6!}{1! \cdot 2! \cdot 3!} = 60$.

Permutations of objects with some alike

Suppose given a collection of n objects containing k subsets of objects in which the objects in each subset are identical and objects in different subsets are not identical. Then the number of different permutations of all n objects is

$$\frac{n!}{r_1! \cdot r_2! \ \cdots r_k!},$$

where r_1 is the number of objects in the first subset, r_2 is the number of objects in the second subset, etc.

Note that if you make the collection of objects into a set, the set has k elements in it.

Note that for a subset of size 1, we have $1! = 1$, so this formula is a generalization of the previous one.

Permutations of objects with some alike

Example How many words can be made from rearrangements of the letters of the word BOOKKEEPER?

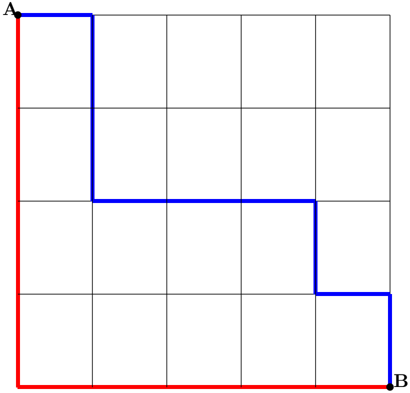
$$\frac{10!}{1! \cdot 3! \cdot 2! \cdot 2! \cdot 1! \cdot 1!} = 151,200.$$

There are 10 letters in BOOKKEEPER. In alphabetical order, B $\leftrightarrow$ 1, E $\leftrightarrow$ 3, K $\leftrightarrow$ 2, O $\leftrightarrow$ 2, P $\leftrightarrow$ 1, R $\leftrightarrow$ 1.

Note that the total number of letters is the sum of the multiplicities of the distinct letters: 10=1+3+2+2+1+1.

Taxi cab geometry

In how many ways can a taxi drive from A to B, going the least possible number of blocks (nine)?



Two possible routes — SSSSEEEEE in red and ESSEEEESES in blue — are shown.

Taxi cab geometry

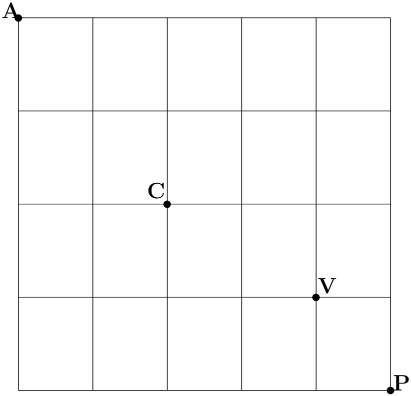
To go using the least number of blocks, the cab must always go South (S) or East (E), and in total must use 4 S's and 5 E's. Any rearrangement of SSSSEEEEE gives a valid route, and there are

$$\frac{9!}{4!5!}$$

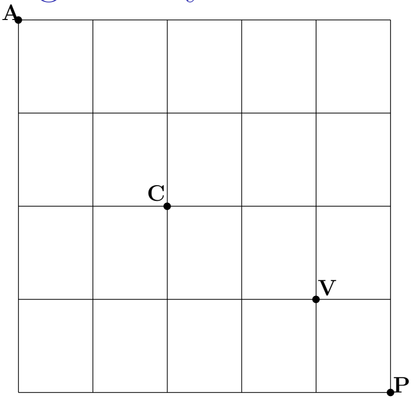
such rearrangements.

Taxi cab geometry

Example A streetmap of Mathville is given below. You arrive at the Airport at A and wish to take a taxi to Pascal's house at P. The taxi driver, being an honest sort, will take a route from A to P with no backtracking, always traveling south or east.



Taxi cab geometry

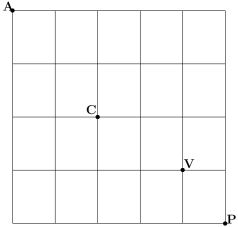


(a) How many such routes are possible from A to P?

You need to go 4 blocks south and 5 blocks east for a total of 9 blocks so the number of routes is

$$\frac{9!}{4! \cdot 5!} = \frac{9 \cdot 8 \cdot 7 \cdot 6}{4 \cdot 3 \cdot 2 \cdot 1} = 9 \cdot 2 \cdot 7 = 126.$$

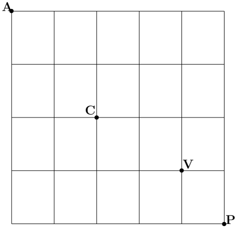
Taxi cab geometry



(b) If you insist on stopping off at the Combinatorium at C, how many routes can the taxi driver take from A to P?

This is really two taxicab problems combined with the Multiplication Principle. The answer, in words, is 'the number of paths from A to C' times 'the number of paths from C to P'. The first is $\dfrac{4!}{2! \cdot 2!} = 6$ and the second is $\dfrac{5!}{2! \cdot 3!} = 10$ so the answer is $6 \cdot 10 = 60$.

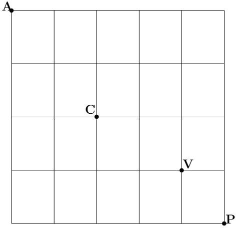
Taxi cab geometry



(c) If wish to stop off at both the combinatorium at C and the Vennitarium at V, how many routes can your taxi driver take?

This is three taxicab problems. The answer, in words, is 'the number of paths from A to C' times 'the number of paths from C to V' times 'the number of paths from V to P. The first is $\dfrac{4!}{2! \cdot 2!} = 6$, the second is $\dfrac{3!}{1! \cdot 2!} = 3$ and the third is $\dfrac{2!}{1! \cdot 1!} = 2$ so the answer is $6 \cdot 3 \cdot 2 = 36$.

Taxi cab geometry

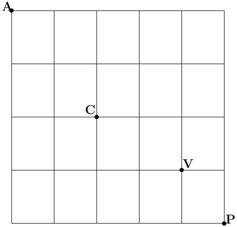


(d) If you wish to stop off at either C or V(at least one), how many routes can the taxi driver take?

This problem involves both taxis and the Inclusion-Exclusion Principle. Suppose C denote the set of all paths from A to P that go through C and that V denotes the set of all paths from A to P that go through V.

The number we want is $n(C \cup V)$ since $C \cup V$ is the set of all paths which go through C or V.

Taxi cab geometry



Suppose C denotes the set of all paths from A to P that go through C and that V denotes the set of all paths from A to P that go through V.

The number we want is $n(C \cup V)$ since $C \cup V$ is the set of all paths which go through C or V.

We have already computed $n(C) = 60$. For $n(V)$ we have

$$n(V) = \frac{7!}{3! \cdot 4!} \cdot \frac{2!}{1! \cdot 1!} = \frac{7 \cdot 6 \cdot 5}{6} \cdot 2 = 70.$$

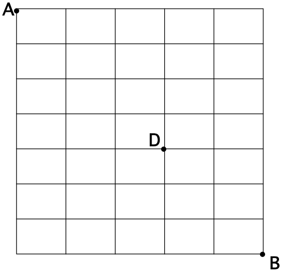
We still need $n(C \cap V)$ but $C \cap V$ is the set of all paths which go through both C and V and we have already computed this: $n(C \cap V) = 36$.

Hence

$$n(C \cup V) = 60 + 70 - 36 = 94$$

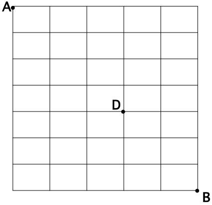
Taxi cab geometry

Example Christine, on her morning run, wants to get from point A to point B.



(a)How many routes with no backtracking can she take?
(b) How many of those routes go through the point D?
(c) If Christine wants to avoid the Doberman at D, how many routes can she take?

Taxi cab geometry



(a)How many routes with no backtracking can she take?
(b) How many of those routes go through the point D?
(c) If Christine wants to avoid the Doberman at D, how many routes can she take?

(a) $\dfrac{(5+7)!}{5!\cdot 7!}$

(b) $\dfrac{(3+4)!}{3!\cdot 4!}\cdot\dfrac{(2+3)!}{2!\cdot 3!}$

(c) $\dfrac{(5+7)!}{5!\cdot 7!}-\left(\dfrac{(3+4)!}{3!\cdot 4!}\cdot\dfrac{(2+3)!}{2!\cdot 3!}\right)$