

Combinations

Example Five friends, Alan, Cassie, Maggie, Seth and Roger, have won 3 tickets for a concert. They can't afford two more tickets. In how many ways can they choose three people from among the five to go?

Here (from last time) is the list of all the ways of choosing three people, when the order of selection matters:

<i>AMC</i>	<i>AMS</i>	<i>AMR</i>	<i>ACS</i>	<i>ACR</i>
<i>ACM</i>	<i>ASM</i>	<i>ARM</i>	<i>ASC</i>	<i>ARC</i>
<i>CAM</i>	<i>MAS</i>	<i>MAR</i>	<i>CAS</i>	<i>CAR</i>
<i>CMA</i>	<i>MSA</i>	<i>MRA</i>	<i>CSA</i>	<i>CRA</i>
<i>MAC</i>	<i>SAM</i>	<i>RAM</i>	<i>SAC</i>	<i>RCA</i>
<i>MCA</i>	<i>SMA</i>	<i>RMA</i>	<i>SCA</i>	<i>RAC</i>
<i>ASR</i>	<i>MSR</i>	<i>MCR</i>	<i>MCS</i>	<i>CRS</i>
<i>ARS</i>	<i>MRS</i>	<i>MRC</i>	<i>MSC</i>	<i>CSR</i>
<i>SAR</i>	<i>SMR</i>	<i>RMC</i>	<i>CMS</i>	<i>RCS</i>
<i>SRA</i>	<i>SRM</i>	<i>RCM</i>	<i>CSM</i>	<i>RSC</i>
<i>RSA</i>	<i>MRS</i>	<i>CRM</i>	<i>SMC</i>	<i>SCR</i>
<i>RAS</i>	<i>MSR</i>	<i>CMR</i>	<i>SCM</i>	<i>SRC</i>

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Now order of selection *doesn't* matter — AMC is the same as ACM — so these sixty possibilities bunch up in groups of 6, with everything in the same group being the same.

That leaves 10 different possibilities:

<i>AMC</i>	<i>AMS</i>	<i>AMR</i>	<i>ACS</i>	<i>ACR</i>
<i>ASR</i>	<i>MSR</i>	<i>MCR</i>	<i>MCS</i>	<i>CRS</i>

With order mattering, there were $\mathbf{P}(5, 3)$ possibilities.

With order not mattering, we have overcounted by a factor of $6 = 3!$ (one for each of the ways or putting an order on three people), so the right count is

$$\frac{60}{3!} = \frac{\mathbf{P}(5, 3)}{3!} = \frac{5!}{2!3!}.$$

Combinations

We have listed all **Combinations** of the five friends taken 3 at a time. The number of such combinations is denoted by $C(5, 3)$.

This is the same as listing all the subsets of size 3 of the set $\{A, C, M, R, S\}$

Definition A **Combination** of n objects taken r at a time is a selection of r objects taken from among the n . The order in which the objects are chosen does not matter.

The key characteristics of a combination are

1. A combination selects elements from a single set.
2. Repetitions are not allowed.
3. The order in which the selected elements are arranged is **not** significant.

Combinations

The **number of such combinations** is denoted by the symbol $\mathbf{C}(n, r)$ or $\binom{n}{r}$. We have

$$\begin{aligned}\mathbf{C}(n, r) &= \binom{n}{r} = \frac{\mathbf{P}(n, r)}{r!} = \\ &= \frac{n \cdot (n - 1) \cdot \dots \cdot (n - r + 1)}{r \cdot (r - 1) \cdot (r - 2) \cdot \dots \cdot 1} = \\ &= \frac{n!}{r! \cdot (n - r)!}\end{aligned}$$

Example Evaluate $\mathbf{C}(10, 3)$.

$$\mathbf{C}(10, 3) = \frac{10!}{3! \cdot 7!} = \frac{10 \cdot 9 \cdot 8}{3 \cdot 2 \cdot 1} = 120$$

Combinations

Example How many ways are there to choose 7 people from a class of 40 students in order to make a team for Bookstore Basketball?

$$C(40, 7) = \frac{40!}{7! \cdot 33!} = 18,643,560$$

Example In a soccer tournament with 15 teams, each team must play each other team exactly once. How many matches must be played?

$$C(15, 2) = \frac{15!}{2! \cdot 13!} = \frac{15 \cdot 14}{2 \cdot 1} = 15 \cdot 7 = 105$$

Combinations

Example A poker hand consists of five cards dealt at random from a standard deck of 52. How many different poker hands are possible?

$$C(52, 5) = \frac{52!}{5! \cdot 47!} = 2,598,960$$

Example A standard deck of cards consists of 13 hearts, 13 diamonds, 13 spades and 13 clubs. How many poker hands consist entirely of clubs?

$$C(13, 5) = \frac{13!}{5! \cdot 8!} = 1,287$$

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Example How many poker hands consist of red cards only?

There are 26 red cards so $C(26, 5) = \frac{26!}{5! \cdot 21!} = 65,780$

Example How many poker hands consist of 2 kings and 3 queens?

There are 4 kings and 4 queens. We can select 2 kings in $C(4, 2)$ ways and we can select 3 queens in $C(4, 3)$ ways. We can distinguish kings from queens so the answer is $C(4, 2) \cdot C(4, 3) = 6 \cdot 4 = 24$.

Combinations

Example (Quality Control) A factory produces light bulbs and ships them in boxes of 50 to their customers. A quality control inspector checks a box by taking out a sample of size 5 and checking if any of those 5 bulbs are defective. If at least one defective bulb is found the box is not shipped, otherwise the box is shipped. How many different samples of size five can be taken from a box of 50 bulbs?

$$C(50, 5) = 2,118,760.$$

Example If a box of 50 light bulbs contains 20 defective light bulbs and 30 non-defective light bulbs, how many samples of size 5 can be drawn from the box so that all of the light bulbs in the sample are good?

$$C(30, 5) = 142,506.$$

Problems using a mixture of counting principles

Example How many poker hands have at least two kings?

There are $C(4, 2)$ ways to get 2 kings and $C(48, 3)$ ways to fill out the hand. Hence there are $C(4, 2) \cdot C(48, 3)$ hands with exactly 2 kings. There are $C(4, 3) \cdot C(48, 2)$ hands with exactly 3 kings and there are $C(4, 4) \cdot C(48, 1)$ hands with exactly 4 kings. Hence there are

$C(4, 2) \cdot C(48, 3) + C(4, 3) \cdot C(48, 2) + C(4, 4) \cdot C(48, 1)$
hands with at least two kings. The number is
 $6 \cdot 17,296 + 4 \cdot 1,128 + 1 \cdot 48 = 108,336$.

Problems using a mixture of counting principles

Example In the Notre Dame Juggling club, there are 5 graduate students and 7 undergraduates. Student Activities will fund 5 people to attend, as long as at least three are undergraduates. In how many ways can 5 people be chosen to go to the performance so that the funding will be granted?

Break the problem up, by number of undergraduates chosen to attend. Three undergraduates: $C(7, 3) \cdot C(5, 2)$; Four undergraduates: $C(7, 4) \cdot C(5, 1)$; Five undergraduates: $C(7, 5) \cdot C(5, 0)$. The number is $35 \cdot 10 + 35 \cdot 5 + 21 \cdot 1 = 546$.

Remark: $C(7, 3) \cdot C(9, 2) = 1,260$. Why is this NOT the right answer?

Problems using a mixture of counting principles

Example Gino's Pizza Parlor offers 3 three types of crust, 2 types of cheese, 4 vegetable toppings and 3 meat toppings. Pat always chooses one type of crust, one type of cheese, 2 vegetable toppings and two meat toppings. How many different pizzas can Pat create?

Pat's choices are independent so

$$\mathbf{C(3, 1) \cdot C(2, 1) \cdot C(4, 2) \cdot C(3, 2) = 3 \cdot 2 \cdot 6 \cdot 3 = 108.}$$

Example How many subsets of a set of size 5 have at least 4 elements?

$$\mathbf{C(5, 4) + C(5, 5).}$$

Special Cases and Formulas

- ▶ It is immediate from the formula $\mathbf{C}(n, k) = \frac{n!}{k!(n-k)!}$ that $\mathbf{C}(n, k) = \mathbf{C}(n, n - k)$ — choosing k things from a set on n to be “in” is the same as choosing $n - k$ things to be “out”
- ▶ $\mathbf{C}(n, 0) = 1$ — there is exactly one subset with zero elements in a set. For the formula to always hold, we want $\frac{n!}{0!(n-0)!} = 1$, so we define $0! = 1$.
- ▶ $\mathbf{C}(n, 1) = \frac{n!}{(n-1)!} = n$ — there are n one-element subsets in a set with n elements.
- ▶ $\mathbf{C}(n, n) = 1$.

How many subsets does a set have?

A set of size 1, say $\{A\}$, has two subsets: \emptyset and $\{A\}$

A set of size 2, say $\{A, B\}$, has four subsets: \emptyset , $\{A\}$, $\{B\}$ and $\{A, B\}$

A set of size 3, say $\{A, B, C\}$, has eight subsets: \emptyset , $\{A\}$, $\{B\}$, $\{C\}$, $\{A, B\}$, $\{A, C\}$, $\{B, C\}$ and $\{A, B, C\}$

A set of size n has 2^n subsets — we can choose a subset by going through each element in turn, and deciding whether it is in the subset or not. By the multiplication principle this experiment has $2 \times 2 \times \dots \times 2 = 2^n$ possible outcomes.

Also, a set of size n has $\mathbf{C}(n, 0)$ subsets of size 0, $\mathbf{C}(n, 1)$ subsets of size 1, $\mathbf{C}(n, 2)$ subsets of size 2, etc., so $\mathbf{C}(n, 0) + \mathbf{C}(n, 1) + \mathbf{C}(n, 2) + \dots + \mathbf{C}(n, n - 1) + \mathbf{C}(n, n)$ subsets in all

How many subsets does a set have?

$$2^n = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \binom{n}{3} + \cdots + \binom{n}{n}$$

$$2^n = \mathbf{C}(n, 0) + \mathbf{C}(n, 1) + \mathbf{C}(n, 2) + \mathbf{C}(n, 3) + \cdots + \mathbf{C}(n, n)$$

Example A set has ten elements. How many of its subsets have at least two elements?

$\mathbf{C}(10, 2) + \mathbf{C}(10, 3) + \mathbf{C}(10, 4) + \mathbf{C}(10, 5) + \mathbf{C}(10, 6) + \mathbf{C}(10, 7) + \mathbf{C}(10, 8) + \mathbf{C}(10, 9) + \mathbf{C}(10, 10)$. To actually compute this number it is easier to compute

$$2^{10} - (\mathbf{C}(10, 0) + \mathbf{C}(10, 1)) = 1024 - (1 + 10) = 1013$$

Example How many tips could you leave at a restaurant, if you have a half-dollar, a one dollar coin, a two dollar note and a five dollar note?

You can leave any subset of your money. You have 4 items so there are $2^4 = 16$ possibilities.

The Binomial Theorem

How does this pattern continue?

- ▶ $x + y = x + y$
- ▶ $(x + y)^2 = x^2 + 2xy + y^2$
- ▶ $(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$
- ▶ $(x + y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$

The Binomial theorem says that for any positive integer n and any two numbers x and y , we have

$$(x + y)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \cdots + \binom{n}{n-1}xy^{n-1} + \binom{n}{n}y^n$$

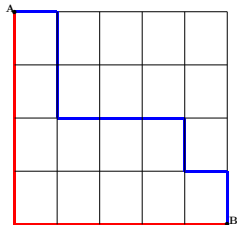
Example If I fully multiply out $(x + y)^{11}$, what's the term involving x^4y^7 ? From the binomial theorem it is $\binom{11}{4} = 330$.

Taxi Cab Geometry revisited

Recall that the number of taxi cab routes (always traveling south or east) from A to B is the number of different rearrangements of the sequence SSSSEEEEE which is

$$\frac{9!}{4!5!} = \mathbf{C(9, 4)} = \mathbf{C(9, 5)}.$$

Sequences SSSSEEEEE (in red) and ESSEEESES (in blue) are shown below.



The number of routes equals the number of ways to choose 4 objects from a set of 9 objects, because to determine a route we can start with nine blank slots, and pick 4 of them to be “S”’s (then the remaining 5 slots are forced to be “E”’s)