Partitions

If we wish to divide a set of size n into disjoint subsets, there are many ways to do this.

Example Six friends Alan, Cassie, Maggie, Seth, Roger and Beth have volunteered to help at a fund-raising show. One of them will hand out programs at the door, two will run a refreshments stand and three will help guests find their seats. In assigning the friends to their duties, we need to divide or partition the set of 6 friends into disjoint subsets of 3, 2 and 1. There are a number of different ways to do this, a few of which are listed below:

Partitions

Prog.	Refr.	Usher
A	CM	SRB
C	AS	MRB
M	CB	ASR
B	SR	ACM
R	CM	SAB

This is not a complete list, it is not difficult to think of other possible partitions. However, we know from experience that it is easier to count the number of such partitions by using our counting principles than it is by listing all of them. We can solve this problem easily by breaking the task into steps

Partitions

Step 1: choose three of the friends to serve as ushers $\left(\mathbf{C}(6,3) = \dfrac{6!}{3! \cdot 3!} \text{ ways}\right)$
Step 2: choose two of the remaining friends to run the refreshment stand $\left(\mathbf{C}(3,2) = \dfrac{3!}{2! \cdot 1!} \text{ ways}\right)$
Step 3: choose one of the remaining friends to hand out programs $\left(\mathbf{C}(1,1) = \dfrac{1!}{1! \cdot 0!} \text{ ways}\right)$

Thus using the multiplication principle, we find that the number of ways that we can split the group of 6 friends into sets of 3, 2 and 1 is

$$\mathbf{C}(6,3) \cdot \mathbf{C}(3,2) \cdot \mathbf{C}(1,1) = \frac{6!}{3! \cdot 3!} \cdot \frac{3!}{2! \cdot 1!} \cdot \frac{1!}{1!0!} = \frac{6!}{3!\cancel{3!}} \cdot \frac{\cancel{3!}}{2!\cancel{1!}} \cdot \frac{\cancel{1!}}{1!0!} = \frac{6!}{3!2!1!}$$

since $0! = 1 = 1!$ we see the answer is 60.

Partitions

This example was about *partitions*.

A set S is **partitioned** into k nonempty subsets $A_1, A_2, \ldots, A_k$ if:

1. Every pair of subsets in disjoint: that is $A_i \cap A_j = \emptyset$ if $i \neq j$.
2. $A_1 \cup A_2 \cup \cdots \cup A_k = S$.

The partition described above is **ordered**: swapping A_1 and A_2 gives a different partition. Ordered partitions come up when different subsets of the partition have characteristics that distinguish then from each other. (As in the example about handing out programs, serving refreshments, and ushing)

The number of ordered partitions

The number of ways to partition a set with n elements into k subsets $A_1, \ldots, A_k$ with A_i having r_i elements is

$$\binom{n}{r_1} \cdot \binom{n-r_1}{r_2} \cdots \binom{n-r_1-\ldots-r_{k-1}}{r_k} = \frac{n!}{r_1! \cdot r_2! \cdots r_k!}$$

There's a special symbol for this: when $n = r_1 + \ldots + r_k$,

$$\boxed{\binom{n}{r_1, r_2, \ldots, r_k} = \frac{n!}{r_1! r_2! \ldots r_k!}}$$

Note 1: The elements in the individual subsets, $A_1, \ldots,$ A_r are not ordered!

Note 2: Partitioning a set with n elements into two sets of sizes r_1 and $n - r_1$ is *exactly* the same as selecting r_1 elements from the set (put these in the first subset, put the rest in the second subset). Hence $\binom{n}{r_1, n-r_1} = \frac{n}{r_1!(n-r_1)!} = \mathbf{C}(n, r_1)$

Ordered Partitions

Example A In how many ways can the group of six friends Alan, Cassie, Maggie, Seth, Roger and Beth, be assigned to three groups of two where two are assigned to hand out programs, two are assigned to the refreshments stand and two are assigned as ushers?

Since each person is assigned to exactly one group, this scheme partitions the set of 6 friends. The subsets are distinguished (ordered) because one is assigned to programs, one to refreshments and one to usher. Hence the answer is $\binom{6}{2,2,2} = \frac{6!}{(2!)^3} = \frac{720}{8} = 90$.

Ordered Partitions

Example In how many ways can a set of ten people be divided into groups of five, three and two?

$$\binom{10}{5,3,2} = \frac{10!}{5!\cdot 3!\cdot 2!} = \frac{10\cdot 9\cdot 8\cdot 7\cdot 6}{12} =$$

$$\frac{10\cdot 9\cdot 8\cdot 7}{2} = 5\cdot 9\cdot 8\cdot 7 = 72\cdot 35 = 2,520\ .$$

Example Evaluate $\binom{7}{3,2,2}$.

$$\binom{7}{3,2,2} = \frac{7!}{3!\cdot 2!\cdot 2!} = \frac{5040}{24} = 210$$

Ordered Partitions

Example A group of 12 new hires at the Electric Car Company will be split into three groups. Four will be sent to Dallas, three to Los Angeles and five to Portland. In how many ways can the group of new hires be divided in this way?

This is an ordered partition problem since the entire set of 12 new hires is divided into 3 disjoint subsets which can be distinguished. Hence the answer is $\binom{12}{4,3,5} = \frac{12!}{5! \cdot 4! \cdot 3!} =$

$$\frac{12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6}{4 \cdot 3 \cdot 2 \cdot 3 \cdot 2} = \frac{11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6}{2 \cdot 3 \cdot 2} =$$

$$\frac{11 \cdot 10 \cdot 9 \cdot 8 \cdot 7}{2} = 11 \cdot 10 \cdot 9 \cdot 4 \cdot 7 = 27,720$$

Ordered Partitions

Example A pre-school teacher will split her class of 15 students into three groups with five students in each group. One group will color, a second group will play in the sand box and the third group will nap. In how many ways can the teacher form the groups for coloring, sand box play and napping?

$$\binom{15}{5,5,5} = \frac{15!}{(5!)^3} = 756,756.$$

Unordered Partitions

A partition is **unordered** when no distinction is made between subsets of the same size (the order of the subsets does not matter).

We use the "overcounting" principle to find a formula for the number of unordered partitions.

Example Our group of 6 friends Alan, Cassie, Maggie, Seth, Roger and Beth have signed up to distribute fliers in the neighborhood. The person who hired them doesn't care how they do this but wants two people in each group. Alan wants to know how many ways they can divide up. In particular the six pairings shown next give us the same unordered partition and is counted only as one such unordered partition or pairing.

Unordered Partitions

AS	CM	RB	The above single unordered partition would have counted as six different ordered partitions if we had a different assignment for each group.
CM	AS	RB	
AS	RB	CM	
CM	RB	AS	
RB	AS	CM	
RB	CM	AS	

Likewise each unordered partition into three sets of two gives rise to 3! ordered partitions and we can calculate the number of unordered partitions by dividing the number of ordered partitions by 3!.

Hence a set with 6 elements can be partitioned into 3 unordered subsets of 2 elements in

$$\frac{1}{3!}\binom{6}{2,2,2} = \frac{6!}{3!\cdot 2!\cdot 2!\cdot 2!} = \frac{6!}{3!(2!)^3} \text{ ways .}$$

Unordered Partitions

In a similar way, we can derive a formula for the number of unordered partitions of a set.

A set of n elements can be partitioned into k **unordered subsets** of r elements each ($kr = n$) in the following number of ways:

$$\boxed{\frac{1}{k!}\binom{n}{r, r, \ldots, r} = \frac{n!}{k! \cdot r! \cdot r! \cdots r!} = \frac{n!}{k!(r!)^k}.}$$

Example In how many ways can a set with 12 elements be divided into four unordered subsets with three elements in each?

$$\frac{12!}{4! \cdot (3!)^4} = 15,400.$$

Unordered Partitions

Example The draw for the first round of the middleweight division for the Bengal Bouts is about to be made. There are 32 competitors in this division. In how many ways can they be paired up for the matches in the first round?

The answer is $\dfrac{32!}{16! \cdot \left((2!)^{16}\right)} = 191,898,783,962,511,000$

Unordered Partitions

In any unordered partition where k of the subsets have the same number of elements, we must divide the number of ordered partitions by $k!$ in order to get the number of unordered partitions. We won't write down a general formula here, but instead work a few examples.

Example Find the number of partitions of a set of 20 elements into subsets of two, two, two, four, four, three and three. No distinction will be made between subsets except for their size.

The number of partitions is $\dfrac{20!}{2! \cdot 2! \cdot 2! \cdot 4! \cdot 4! \cdot 3! \cdot 3!}$ but these are ordered in that there is a first subset with 2 elements, a second subset with 2 elements and a third subset with 2 elements. There are similar remarks about the 3 and 4 element subsets.

Unordered Partitions

If you don't care about the order, the answer is

$$\frac{20!}{(2! \cdot 2! \cdot 2! \cdot 4! \cdot 4! \cdot 3! \cdot 3!) \cdot 3! \cdot 2! \cdot 2!} =$$

$$\frac{20!}{(2!)^5 \cdot (3!)^3 \cdot (4!)^2} = \frac{20!}{2 \cdot 6 \cdot \left((2!)^2 \cdot 3! \cdot 4!\right)^2} =$$

$$\frac{20!}{12 \cdot (4 \cdot 6 \cdot 24)^2} = \frac{2 \cdot 20!}{(24)^5} = 611,080,470,000$$

Unordered Partitions

Example A math teacher wishes to split a class of thirty students into groups. All groups will work on the same problem. Five groups will have four students, two groups will have three students and two groups will have two students. In how many ways can the teacher assign students to the groups?

$$\frac{30!}{\left((4!)^5 \cdot (3!)^2 \cdot (2!)^2\right) \cdot 5! \cdot 2! \cdot 2!} =$$

$$\frac{30!}{5! \cdot (4!)^5 \cdot (3!)^2 \cdot (2!)^4} = 481,947,949,543,123,000,000$$