

Bayes' Theorem

Now we look at how we can use information about conditional probabilities to calculate reverse conditional probabilities — i.e., how we calculate $\mathbf{P}(A|B)$ when we know $\mathbf{P}(B|A)$ (and some other things). We will generally solve these problems with tree diagrams, but we will also see a formula that says what's going on algebraically.

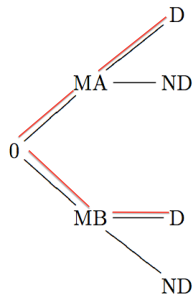
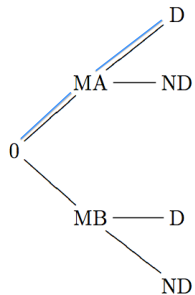
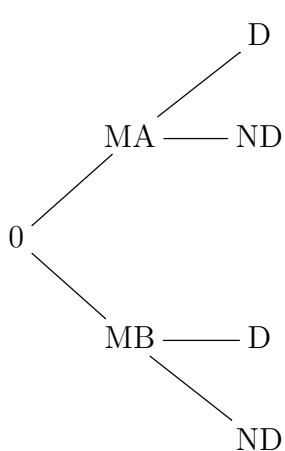
Example Suppose that a factory has two machines, Machine A and Machine B, both producing iPhone touch screens. 40% of production is from Machine A and 60% is from Machine B. 10% of the touch screens produced by Machine A are defective and 5% of those from Machine B are.

If I randomly choose a touch screen produced in the factory and find that it is defective, what is the probability that it came from Machine A?

Factory example

We can draw a tree diagram representing the information we are given. If we choose a touch screen at random from those produced in the factory, we let MA be the event that it came from Machine A and let MB be the event that it came from Machine B. We let D denote the event that the touch screen is defective and let ND denote the event that it is not defective. Fill in the appropriate probabilities on the tree diagram on the left on the next page.

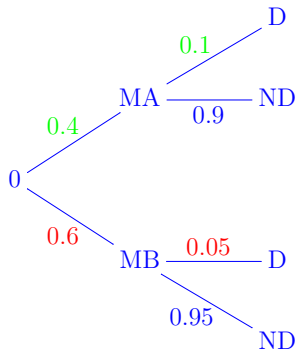
Factory example



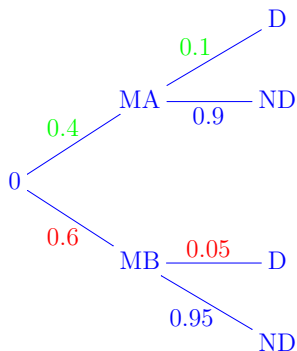
Factory example

We can now calculate $\mathbf{P}(MA|D) = \frac{\mathbf{P}(MA \cap D)}{\mathbf{P}(D)}$ =

$\frac{\mathbf{P}(MA \cap D)}{\mathbf{P}(D|MA) \cdot \mathbf{P}(MA) + \mathbf{P}(D|MB) \cdot \mathbf{P}(MB)}$. Note the event D is shown in red above and the event $MA \cap D$ is shown in blue.



Factory example



$$\begin{aligned} \mathbf{P}(MA|D) &= \frac{\mathbf{P}(MA \cap D)}{\mathbf{P}(D)} = \\ &= \frac{\mathbf{P}(D|MA) \cdot \mathbf{P}(MA) + \mathbf{P}(D|MB) \cdot \mathbf{P}(MB)}{0.4 \cdot 0.1 + 0.6 \cdot 0.05} = \frac{0.04}{0.04 + 0.03} = \frac{0.04}{0.07} = \frac{4}{7} \end{aligned}$$

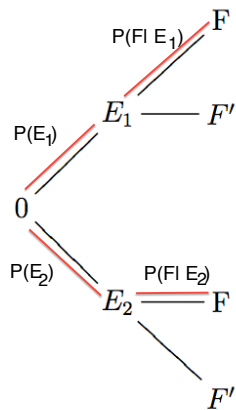
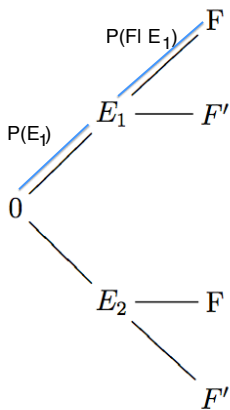
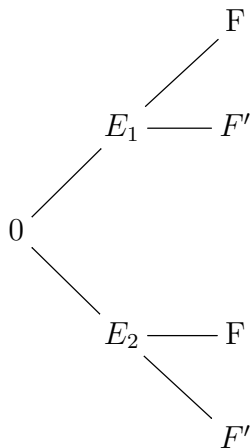
Bayes' Theorem

Let E_1 and E_2 be mutually exclusive events ($E_1 \cap E_2 = \emptyset$) whose union is the sample space, i.e. $E_1 \cup E_2 = S$. Let F be an event in S for which $\mathbf{P}(F) \neq 0$. Then

$$\mathbf{P}(E_1|F) = \frac{\mathbf{P}(E_1 \cap F)}{\mathbf{P}(F)} = \frac{\mathbf{P}(E_1 \cap F)}{\mathbf{P}(E_1 \cap F) + \mathbf{P}(E_2 \cap F)} = \frac{\mathbf{P}(E_1)\mathbf{P}(F|E_1)}{\mathbf{P}(E_1)\mathbf{P}(F|E_1) + \mathbf{P}(E_2)\mathbf{P}(F|E_2)}.$$

Note that if we cross-classify outcomes in the sample space according to whether they belong to E_1 or E_2 and whether they belong to F or F' , we get a tree diagram as above from which we can calculate the probabilities.

Bayes' Theorem



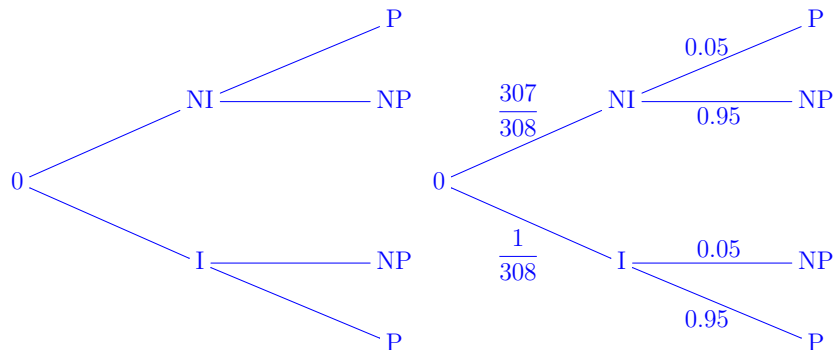
Predictive value of diagnostic test

Bayes' theorem allows us to gain insight about the accuracy of tests for diseases and drugs.

Example: Around one million people in the US (population around 308 million) have a certain particularly nasty condition, condition X. A test exists for X, which is 95% accurate — the test correctly identifies the presence of X 95% of the time, and correctly identifies the absence of X 95% of the time.

Being a hypochondriac, I have myself tested for X, and the test comes back positive. What is the probability that I have X? That is, letting P denote the event that a person chosen at random from the population tests positive, and letting I denote the event that a person chosen at random has X, what is $\mathbf{P}(I|P)$?

Predictive value of diagnostic test

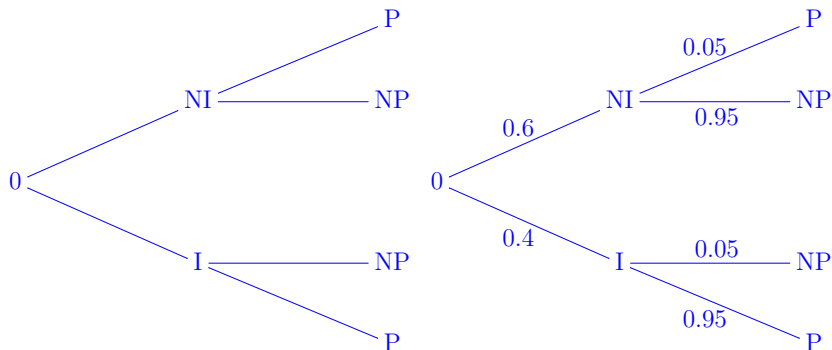


$$\mathbf{P}(I|P) \approx \frac{\frac{1}{308} \cdot 0.95}{\frac{307}{308} \cdot 0.05 + \frac{1}{308} \cdot 0.95} \approx \frac{0.00308}{0.0498 + 0.00308} \approx$$

$$\frac{0.00308}{0.0529} \approx 0.0583 = 5.83\%$$

Predictive value of diagnostic test

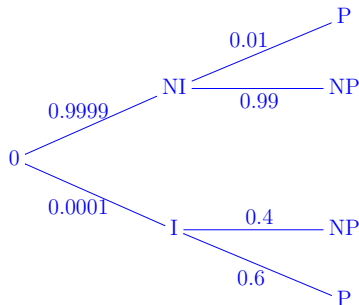
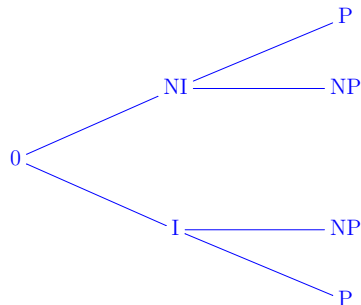
In country C, 40% of the residents have condition X.
Suppose a random resident of Country C tests positive for X. What is the probability that the person actually has X?



$$\mathbf{P}(I|P) = \frac{0.4 \cdot 0.95}{0.6 \cdot 0.05 + 0.4 \cdot 0.95} \approx 93\%$$

Predictive value of diagnostic test

Example A test for Lyme disease is 60% accurate when a person has the disease and 99% accurate when a person does not have the disease. In Country Y, 0.01% of the population has Lyme disease. What is the probability that a person chosen randomly from the population who test positive for the disease actually has the disease?



$$\mathbf{P}(I|P) = \frac{0.0001 \cdot 0.06}{0.9999 \cdot 0.01 + 0.0001 \cdot 0.06} \approx 0.006.$$

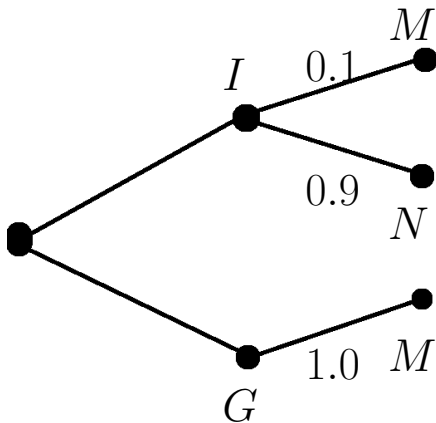
A legal example

A crime has been committed and the only evidence is a blood spatter that could only have come from the perpetrator. The chance of a random individual having the same blood type as that of the spatter is 10%. Joe has been arrested and charged. The trial goes as follows.

Prosecutor: Since there is only a 10% chance that Joe's blood would match, there is a 90% chance that Joe did it. That's good enough for me.

Defence Lawyer: There are two hundred people in the neighborhood who could have done the crime. Twenty of them will have the same blood type as the sample. Hence the chances that Joe did it are $\frac{1}{20} = 5\%$ so there is a 95% chance that Joe is innocent. That's good enough for me.

The ghost of the Reverend Thomas Bayes: You're all nuts!



If $\mathbf{P}(I) = x$ and so $\mathbf{P}(G) = 1 - x$ then

$$\mathbf{P}(I|M) = \frac{0.1 \cdot x}{0.1 \cdot x + 1 \cdot (1 - x)} = \frac{0.1x}{1 - 0.9x}$$

$$\text{If } \mathbf{P}(I) = x \text{ then } \mathbf{P}(I|M) = \frac{0.1x}{1 - 0.9x}$$

If you are initially certain that Joe is guilty, then $x = 0$ and after seeing the evidence you are still certain, $\mathbf{P}(I|M) = 0$.

If you are initially certain that Joe is innocent, then $x = 1$ and after seeing the evidence you are still certain, $\mathbf{P}(I|M) = 1$.

If you think that Joe there is a 40% change that Joe is guilty, then $x = 0.4$ and after seeing the evidence $\mathbf{P}(I|M) = 0.0625$.

If you suspect that the police searched a blood-type database until they came up with a name in the neighborhood, then you might initially think that $x = P(I) = \frac{19}{20} = 95\%$. Now after seeing the evidence Bayes suggests revising to $\mathbf{P}(I|M) = 0.66$.

Some resources

Here's an article on the predictive value of diagnostic tests:

[Doctors flunk quiz on screening-test math](#)

If you are more legally inclined, [here](#) is a discussion of Bayes Theorem as it applies to criminal trials.

A more general Bayes' Theorem

Let E_1, E_2, \dots, E_n be (pairwise) mutually exclusive events such that $E_1 \cup E_2 \cup \dots \cup E_n = S$, where S denotes the sample space. Let F be an event such that $\mathbf{P}(F) \neq 0$, Then

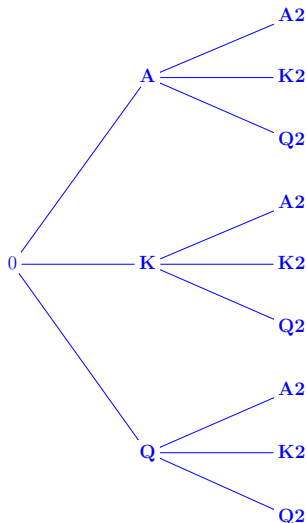
$$\begin{aligned} \mathbf{P}(E_1|F) &= \frac{\mathbf{P}(E_1 \cap F)}{\mathbf{P}(F)} = \\ & \frac{\mathbf{P}(E_1 \cap F)}{\mathbf{P}(E_1 \cap F) + \mathbf{P}(E_2 \cap F) + \dots + \mathbf{P}(E_n \cap F)} = \\ & \frac{\mathbf{P}(E_1)\mathbf{P}(F|E_1)}{\mathbf{P}(E_1)\mathbf{P}(F|E_1) + \mathbf{P}(E_2)\mathbf{P}(F|E_2) + \dots + \mathbf{P}(E_n)\mathbf{P}(F|E_n)} \end{aligned}$$

A more general Bayes' Theorem

Example A pile of 8 playing cards has 4 aces, 2 kings and 2 queens. A second pile of 8 playing cards has 1 ace, 4 kings and 3 queens. You conduct an experiment in which you randomly choose a card from the first pile and place it on the second pile. The second pile is then shuffled and you randomly choose a card from the second pile. If the card drawn from the second deck was an ace, what is the probability that the first card was also an ace?

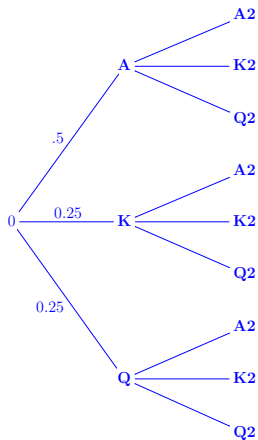
A more general Bayes' Theorem

Let **A** be the event that you draw an ace, **K** the event that you draw a king and **Q** be the event that you draw a queen.



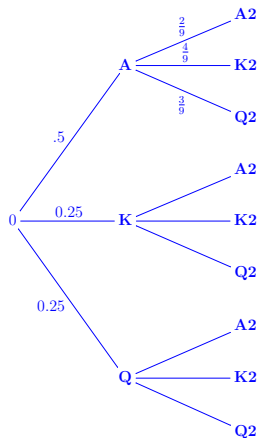
A more general Bayes' Theorem

In the first round there are $4 + 2 + 2 = 8$ cards so the probabilities in the first round are



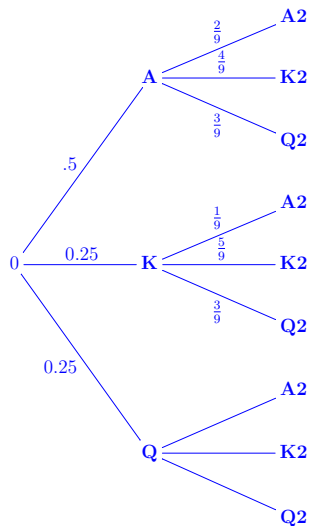
A more general Bayes' Theorem

In the second round there are $1 + 4 + 3 + 1 = 9$ cards and the probabilities are different at the various nodes. If you draw an ace in round 1 the cards are 2 aces, 4 kings and 3 queens so we get



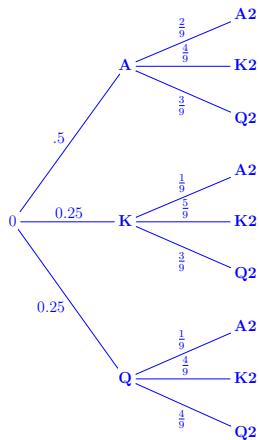
A more general Bayes' Theorem

If you draw a king in round 1 the cards are 1 ace, 5 kings and 3 queens so we get

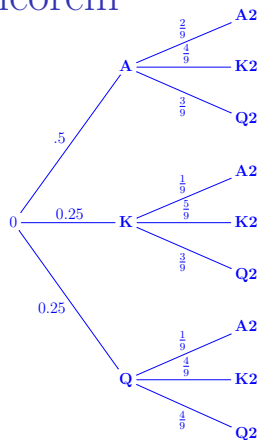


A more general Bayes' Theorem

If you draw a queen in round 1 the cards are 1 ace, 4 kings and 4 queens so we get



A more general Bayes' Theorem



The question asks for

$$\mathbf{P}(A|A2) = \frac{\mathbf{P}(A \cap A2)}{\mathbf{P}(A2)} = \frac{\frac{1}{2} \cdot \frac{2}{9}}{\frac{1}{2} \cdot \frac{2}{9} + \frac{1}{4} \cdot \frac{1}{9} + \frac{1}{4} \cdot \frac{1}{9}} = \frac{2}{3}.$$