Department of Mathematics
University of Notre Dame
Math 10120 – Finite Math
Spring 2017

Name:______________________________

Instructors: Conant/Galvin

Exam 2 Solutions

March 13, 2017

This exam is in two parts on 12 pages and contains 15 problems worth a total of 100 points. You have 1 hour and 15 minutes to work on it. You **may** use a calculator, but **no** books, notes, or other aid is allowed. Be sure to write your name on this title page and put your initials at the top of every page in case pages become detached.

Record your answers to the multiple choice problems on this page. Place an × through your answer to each problem.

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May the odds be ever in your favor!

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Multiple Choice

1. (5 pts.) Two fair dice are rolled. What is the probability that the sum of the numbers that come up is either 4 or 5? ("Fair" means that each of the numbers 1 through 6 are equally likely.)

(a) $\frac{7}{36}$ (b) $\frac{12}{36}$ (c) $\frac{9}{36}$

(d) $\frac{8}{36}$ (e) $\frac{6}{36}$

Answer: The event "sum of numbers is 4 or 5" consists of 7 total outcomes. Specifically, there are 4 ways to sum to 5: $(1,4)$, $(4,1)$, $(2,3)$, and $(3,2)$; and there are 3 ways to sum to 4: $(1,3)$, $(3,1)$, and $(2,2)$. Since the total sample space contains 36 outcomes, the answer is $\frac{7}{36}$.

2. (5 pts.) The space of outcomes for a particular experiment is $S = \{a,b,c,d,e\}$. Suppose $\mathbf{P}(\{a,b,c\}) = \frac{3}{5}$ and $\mathbf{P}(\{c,d,e\}) = \frac{7}{10}$. Find $\mathbf{P}(\{c\})$.

(a) $\frac{3}{10}$ (b) $\frac{1}{10}$ (c) $\frac{13}{15}$

(d) $\frac{21}{50}$ (e) $\frac{4}{10}$

Answer: Use the Principle of Inclusion-Exclusion for probability. Specifically, since $S = \{a,b,c\} \cup \{c,d,e\}$ and $\{a,b,c\} \cap \{c,d,e\} = \{c\}$, the principle says:

$$1 = \mathbf{P}(S) = \mathbf{P}(\{a,b,c\}) + \mathbf{P}(\{b,c,d\}) - \mathbf{P}(\{c\}) = \tfrac{3}{5} + \tfrac{7}{10} - \mathbf{P}(\{c\}).$$

So $\mathbf{P}(\{c\}) = \frac{3}{5} + \frac{7}{10} - 1 = \frac{3}{10}$.

3. (5 pts.) A quiz consists of 8 multiple choice problems, each with 5 possible answers. If a student randomly guesses on each problem, what is the probability they get at least one correct? (Answers are rounded to five decimal places.)

(a) 0.83223 (b) 0.00001 (c) 0.99998

(d) 0.20971 (e) 0.63923

Answer: It is easier to calculate the probability of the complement, which is the probability the student gets all answers incorrect. On any particular problem, the probability of guessing incorrectly is $\frac{4}{5}$. Since the student is guessing independently on all problems, the probability of getting all 8 incorrect is $(\frac{4}{5})^8$. So the probability of getting at least one correct is $1 - (\frac{4}{5})^8 \approx 0.83223$.

4. (5 pts.) Out of 130 total rides at a Six Flags amusement park, 45 have height restrictions, 75 have age restrictions, and 15 have no restrictions. If a ride is selected at random, what is the probability it has a height restriction but not an age restriction?

(a) $\dfrac{40}{130}$ (b) $\dfrac{45}{130}$ (c) $\dfrac{75}{130}$

(d) $\dfrac{5}{130}$ (e) $\dfrac{30}{130}$

Answer: A Venn diagram is helpful for this problem. Let A be the set of rides with age restrictions, let H be the set of rides with height restrictions. The number of rides neither in A nor H (i.e. $n((A \cup H)'))$ is 15. Since the are 130 total rides, there are 115 rides in $A \cup H$. Since $n(A) + n(H) = 120$, there must be 5 rides in $A \cap H$. So the number of rides with height restrictions, but not age restrictions is $45 - 4 = 40$. The probability of selecting one these rides is therefore $\frac{40}{130}$.

5. (5 pts.) E and F are events in a sample space S. Suppose $\mathbf{P}(E \cup F) = 0.8$, $\mathbf{P}(E \cap F) = 0.1$, $\mathbf{P}(E) = 0.5$ and $\mathbf{P}(F) = 0.4$. What is $\mathbf{P}(E|F)$?

(a) 25% (b) 10% (c) 20%

(d) 12.5% (e) 30%

Answer: By definition,

$$\mathbf{P}(E|F) = \frac{\mathbf{P}(E \cap F)}{\mathbf{P}(F)} = \frac{0.1}{0.4} = \frac{1}{4} = 25\%.$$

6. (5 pts.) Suppose we have the following information:

$$\begin{array}{ccc} \mathbf{P}(E) = \frac{1}{2} & \mathbf{P}(F) = \frac{1}{3} & \mathbf{P}(G) = \frac{1}{4} \\ \mathbf{P}(E \cap F) = \frac{1}{6} & \mathbf{P}(E \cap G) = \frac{1}{8} & \mathbf{P}(F \cap G) = \frac{1}{10}. \end{array}$$

Which pair(s) of events are independent?

(a) E and F are independent, and E and G are independent.

(b) E and G are independent, and F and G are independent.

(c) E and F are independent, E and G are independent, and F and G are independent.

(d) E and F are independent, and F and G are independent

(e) There are no pairs of events that are independent.

Answer: Recall that events A and B are independent if and only if $\mathbf{P}(A \cap B) = \mathbf{P}(A) \cdot \mathbf{P}(B)$. So check the pairs:

$$\begin{array}{c} \mathbf{P}(E \cap F) = \frac{1}{6} \text{ and } \mathbf{P}(E) \cdot \mathbf{P}(F) = \frac{1}{2} \cdot \frac{1}{3} = \frac{1}{6} \\ \mathbf{P}(E \cap G) = \frac{1}{8} \text{ and } \mathbf{P}(E) \cdot \mathbf{P}(G) = \frac{1}{2} \cdot \frac{1}{4} = \frac{1}{8} \\ \mathbf{P}(F \cap G) = \frac{1}{10} \text{ and } \mathbf{P}(F) \cdot \mathbf{P}(G) = \frac{1}{3} \cdot \frac{1}{4} = \frac{1}{12} \end{array}$$

So E and F are independent, E and G are independent, but F and G are not independent.

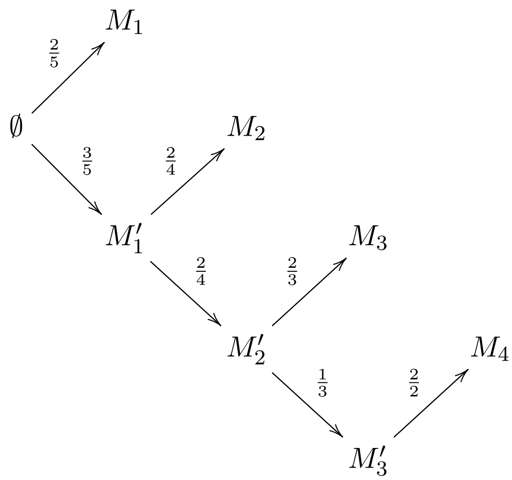
7. (5 pts.) I have five envelopes in front of me; two of them have some money in them, and the other three are empty. I open the envelopes one after another, in random order. I stop as soon as I have found an envelope that has money in it. What is the probability that I stop just after I open the third envelope? (I.e., that I find money for the first time in the third envelope I try?) [Hint: tree diagram.]

(a) $\frac{1}{5}$ (b) $\frac{13}{20}$ (c) $\frac{1}{10}$ (d) $\frac{3}{10}$ (e) $\frac{2}{5}$

Answer: Let M_n be the event of finding money on the n^{th} envelope. The problem asking for the probability that I don't find money in the first envelope *and* I don't find money in the second envelope *and* I do find money in the third envelope, which is $\mathbf{P}(M_1' \cap M_2' \cap M_3)$. To compute this:

$$\mathbf{P}(M_1' \cap M_2' \cap M_3) = \mathbf{P}(M_1') \cdot \mathbf{P}(M_2' | M_1') \cdot \mathbf{P}(M_3 | M_1' \cap M_2') = \frac{3}{5} \cdot \frac{2}{4} \cdot \frac{2}{3} = \frac{1}{5}.$$

Here is the complete tree diagram (note that since only three envelopes have no money, you are guaranteed to find money by at worst the fourth try).



The problem is asking about the branch ending in M_3.

8. (5 pts.) My computer has two fans, that operate independently. The probability that the first is still working after 24 hours of continuous use is 0.78, and the probability that the second is still working after 24 hours of continuous use is 0.67. What is the probability that **exactly one of them** is still working after 24 hours of continuous use?

(a) 0.4048 (b) 0.5226 (c) 0.2574 (d) 0.1474 (e) 0.4774

Answer: There are two distinct cases for exactly one computer to be working:

(1) the first fan is working *and* the second fan is not working; *or*
(2) the first fan is not working *and* the second fan is working.

The probability of the first case is (0.78)(.33). The probability of the second case is (.22)(.67). The answer is the sum of these numbers, which is 0.4048.

9. (5 pts.) A coin is weighted so that when it is tossed, the probability that it comes up Heads is two-thirds, and the probability that it comes up Tails is one-third. It is tossed five times in a row, all tosses independent of each other. Given the information that the last two tosses are both Heads, what is the probability that all five tosses are Heads?

(a) $\frac{8}{27}$ (b) $\frac{32}{243}$ (c) $\frac{1}{8}$

(d) $\frac{1}{32}$ (e) $\frac{8}{243}$

Answer: Let E be the event "all five tosses are Heads" and let F be the event "the last two tosses are Heads." The problem is asking for $\mathbf{P}(E|F)$, which by definition is

$$\frac{\mathbf{P}(E \cap F)}{\mathbf{P}(F)}$$

Note that the E is contained in F (if all five tosses are heads then the last two are Heads). So $E \cap F = E$, which means

$$\mathbf{P}(E|F) = \frac{\mathbf{P}(E)}{\mathbf{P}(F)}$$

Since the probability of heads on any particular toss is $\frac{2}{3}$, and the tosses are independent, the probability of E (five tosses are Heads) is $(\frac{2}{3})^5$. By similar reasoning the probability of F (the last two tosses are Heads) is $(\frac{2}{3})^2$. So the answer is

$$\mathbf{P}(E|F) = \frac{(\frac{2}{3})^5}{(\frac{2}{3})^2} = \left(\frac{2}{3}\right)^3 = \frac{8}{27}.$$

Here's an alternative solution: given the information that the last two tosses are both Heads, we are reducing to the smaller sample space consisting of all possibilities for the first three tosses, and the only way that all five tosses are Heads is if the first three are all Heads. By independence of coin tosses, this probability is $(2/3)^3 = 8/27$.

10. (5 pts.) Six people respond to a survey that includes the question "How many pets do you have in your house?", and their answers to the question are

$$3 \quad 3 \quad 4 \quad 1 \quad 1 \quad 0.$$

Find the mean, median and mode(s) of this data set.

(a) The mean is 2, the median is 2, the modes are 1 and 3.

(b) The mean is 2, the median is 2, there is no mode.

(c) The mean is 2, the medians are 1 and 3, the modes are 1 and 3.

(d) The mean is 2.4, the medians are 1 and 3, there is no mode.

(e) The mean is 2.4, the median is 2, the modes are 1 and 3.

Answer: The mean is

$$\frac{3+3+4+1+1+0}{6} = \frac{12}{6} = 2.$$

In ascending order the responses are:

$$0 \; 1 \; 1 \; 3 \; 3 \; 4.$$

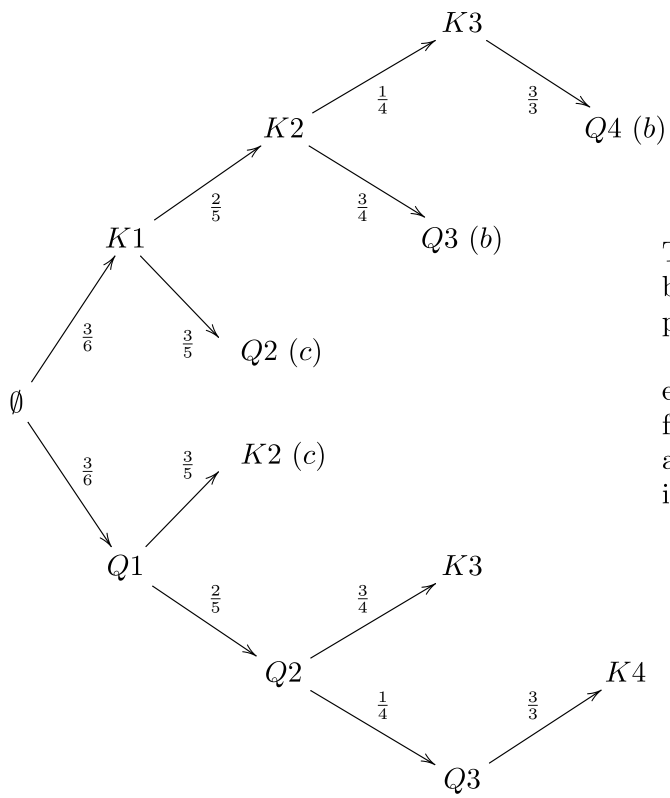
Since there are an even number of data points, the median is the mean of the middle two, i.e. the mean of 1 and 3, which is 2. Finally, both 1 and 3 have the highest frequency and so they are both modes.

Partial Credit

You must show **all of your work** on the partial credit problems to receive full credit! Make sure that your answer is **clearly** indicated. You're more likely to get partial credit for a wrong answer if you explain your reasoning.

11. (10 pts.) An experiment consists of drawing one card at a time without replacement from a deck containing three kings and three queens (six cards in total). You stop drawing cards once you have drawn at least one king and at least one queen (in any order). For this question, fully work out the answers to parts (b) and (c).

(a) Draw a complete tree diagram for this experiment. All branches of the diagram should be labeled with probabilities and events (use "K1" for "first card is a king", et cetera).



The letters (b) and (c) at the end of certain branches are there for reference in the next two parts of the problem.

Notice that since there are only three cards of each kind, you are guaranteed a queen on the fourth draw if first you draw three kings in a row, and you are guaranteed a king on the fourth draw if you first draw three queens in a row.

(b) What is the probability of drawing at least two kings?

Answer: The branches corresponding to this event are the ones labeled (b) above. The sum of their probabilities is: $(\frac{3}{6} \cdot \frac{2}{5} \cdot \frac{1}{4}) + (\frac{3}{6} \cdot \frac{2}{5} \cdot \frac{3}{4}) = \frac{3}{6} \cdot \frac{2}{5} = \frac{1}{5}$

(c) What is the probability of stopping after drawing exactly two cards?

Answer: The branches corresponding to this event are the ones labeled (c) above. The sum of their probabilities is: $(\frac{3}{6} \cdot \frac{3}{5}) + (\frac{3}{6} \cdot \frac{3}{5}) = \frac{3}{5}$

12. (10 pts.) A bag contains 7 blue marbles, 4 red marbles, and 5 white marbles. For this question, you do not need to simplify your answers (i.e. you can use notation for permutations, combinations, and factorials).

(a) Suppose 3 marbles are selected altogether at random. What is the probability that all 3 of them are the same color?

Answer: There are three different cases depending on color. There are $\binom{7}{3}$ ways to choose three blue marbles all at once, there are $\binom{4}{3}$ ways to choose three red marbles all at once, and there are $\binom{5}{3}$ ways to choose three white marbles all at once. Since there are 16 total marbles, there are $\binom{16}{3}$ ways to choose three marbles all at once. So the answer is

$$\frac{\binom{7}{3}+\binom{4}{3}+\binom{5}{3}}{\binom{16}{3}}$$

(b) Suppose 3 marbles are drawn one at a time *with replacement*. What is the probability of drawing red, then white, then blue?

Answer: *With replacement*, the probability of drawing red is $\frac{7}{16}$, and the probability of then drawing white is $\frac{5}{16}$, and then the probability of then drawing blue is $\frac{4}{16}$. So the answer is

$$\frac{7\cdot 5\cdot 4}{16^3}$$

(c) Suppose 3 marbles are drawn one at a time *without replacement*. What now is the probability of drawing red, then white, then blue?

Answer: *Without replacement*, the probability of drawing red is $\frac{7}{16}$, and the probability of then drawing white is $\frac{5}{15}$, and then the probability of then drawing blue is $\frac{4}{14}$. So the answer is

$$\frac{7\cdot 5\cdot 4}{16\cdot 15\cdot 14}$$

13. (10 pts.) An experiment consists of tossing a fair coin 8 times in a row. For this question, fully work out the answers.

(a) What is the probability of the following sequence of tosses: heads, heads, tails, heads, tails, tails, heads, heads?

Answer: This is exactly one outcome out of the total $2^8 = 256$ possibilities. So the answer is $\frac{1}{256}$.

(b) What is the probability of getting at most three heads?

Answer: There are $\binom{8}{0}$ ways of getting exactly 0 heads, $\binom{8}{1}$ ways of getting exactly 1 head, $\binom{8}{2}$ ways of getting exactly 2 heads, and $\binom{8}{3}$ ways of getting exactly 3 heads. So the answer is

$$\frac{\binom{8}{0} + \binom{8}{1} + \binom{8}{2} + \binom{8}{3}}{256} = \frac{1 + 8 + 28 + 56}{256} = \frac{93}{256}$$

(c) What is the probability getting exactly four heads, which all occur consecutively?

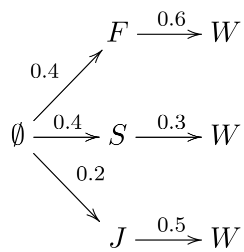
Answer: Count the ways this can happen by observing where in the sequence the four heads occur. If there are four heads then there must be four tails. The four consecutive heads can occur in one of five places (before the tails, right after the first tail, right after the second tail, right after the third tail, or right after the fourth tail). So the answer is

$$\frac{5}{256}$$

14. (10 pts.) In a large calculus class, 40% of the students are first-years, 40% are sophomores and 20% are juniors. Among the first-years, 60% are women and 40% are men. Among the sophomores, 30% are women and 70% are men. Among the juniors, 50% are women and 50% are men. For this question, fully work out the answers.

(a) Let W be the event that a randomly chosen student is a woman. Compute $\mathbf{P}(W)$. [A tree diagram might be useful for this part as well as the next part].

Answer: Let F be the event the student is a first-year, S the event the student is a sophomore, and J the event the student is a junior. The following is a partial tree diagram, in which only the branches ending in W are present (since this is all the problem is concerned with).



Now $\mathbf{P}(W)$ is the sum of the probabilities along three branches ending in W, which is

$$(0.4)(0.5) + (0.4)(0.3) + (0.2)(0.5) = 0.42$$

(b) Given the information that a randomly chosen student is a women, what is the probability that she is a first-year? (That is, compute $\mathbf{P}(F|W)$ where F is the event that a randomly chosen student is a first-year.)

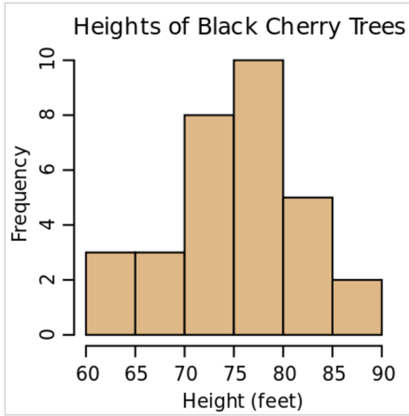
Answer: By definition and the multiplication rule,

$$\mathbf{P}(F|W) = \frac{\mathbf{P}(F \cap W)}{\mathbf{P}(W)} = \frac{\mathbf{P}(F) \cdot \mathbf{P}(W|F)}{\mathbf{P}(W)} = \frac{0.4 \cdot 0.6}{0.42} = \frac{24}{42} = \frac{4}{7}$$

(c) Are the events F and W independent?

Answer: $\mathbf{P}(F) = 0.4$ and $\mathbf{P}(F|W) = \frac{4}{7} \approx 0.5714$. Since these probabilities are different the events are not independent.

15. (10 pts.) The heights of 31 black cherry trees in Fernwood Gardens were measured, and the histogram below shows the results, with categories $[60, 65)$, $[65, 70)$, $[70, 75)$, $[75, 80)$, $[80, 85)$, $[85, 90)$.



(a) Draw up a frequency table of the data in the space below, using the same categories as in the histogram.

Category	Frequency
[60,65)	3
[65,70)	3
[70,75)	8
[75,80)	10
[80,85)	5
[85,90)	2

(b) Estimate the (sample) mean height of a Black Cherry tree from the given data, using the "midpoint rule" for each category. (Fully work out your answer.)

Answer: The midpoints of the intervals are 62.5, 67.5, 72.5, 77.5, 82.5, and 87.5. Using the frequencies for each interval from part (a), the mean estimate is:

$$\frac{(62.5 \cdot 3) + (67.5 \cdot 3) + (72.5 \cdot 8) + (77.5 \cdot 10) + (82.5 \cdot 5) + (87.5 \cdot 2)}{31} \approx 75.242$$

(c) If a tree is chosen at random from among the 31, which is more likely: **A**: that its height is 80 feet of greater, or **B**: that it is less than 70 feet?

Answer: Using the frequency table, there are 7 trees with height 80 or more, and 6 trees with height less than 70. So **A** is more likely.

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