Permutations

**Example** Alan, Cassie, Maggie, Seth and Roger want to take a photo in which three of the five friends are lined up in a row. How many different photos are possible?

<table>
<thead>
<tr>
<th>AMC</th>
<th>AMS</th>
<th>AMR</th>
<th>ACS</th>
<th>ACR</th>
</tr>
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<tbody>
<tr>
<td>ACM</td>
<td>ASM</td>
<td>ARM</td>
<td>ASC</td>
<td>ARC</td>
</tr>
<tr>
<td>CAM</td>
<td>MAS</td>
<td>MAR</td>
<td>CAS</td>
<td>CAR</td>
</tr>
<tr>
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<td>MSA</td>
<td>MRA</td>
<td>CSA</td>
<td>CRA</td>
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<tr>
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<td>SAM</td>
<td>RAM</td>
<td>SAC</td>
<td>RCA</td>
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<tr>
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<td>CSR</td>
</tr>
<tr>
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<td>SMR</td>
<td>RMC</td>
<td>CMS</td>
<td>RCS</td>
</tr>
<tr>
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<td>RCM</td>
<td>CSM</td>
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<tr>
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<td>MRS</td>
<td>CRM</td>
<td>SMC</td>
<td>SCR</td>
</tr>
<tr>
<td>RAS</td>
<td>MSR</td>
<td>CMR</td>
<td>SCM</td>
<td>SRC</td>
</tr>
</tbody>
</table>

60, via an exhaustive (and exhausting!) list
Permutations

Easier, using multiplication principle:

5 options for the person on the left, and

once we’ve chosen who should stand on the left, 4 options for the position in the middle

and once we’ve filled both those positions, 3 options for the person on the right.

This gives a total of $5 \times 4 \times 3 = 60$ possibilities.

We have listed all **Permutations** of the five friends taken 3 at a time.

$$P(5, 3) = 60$$
Permutations

A permutation of \( n \) objects taken \( k \) at a time is an arrangement of \( k \) of the \( n \) objects in a specific order. The symbol for this number is \( P(n, k) \).

Remember:

1. A permutation is an arrangement or sequence of selections of objects from a single set.

2. Repetitions are not allowed. Equivalently the same element may not appear more than once in an arrangement. (In the example above, the photo AAA is not possible).

3. The order in which the elements are selected or arranged is significant. (In the above example, the photographs AMC and CAM are different).
Example Calculate $P(10, 3)$, the number of photographs of 10 friends taken 3 at a time.

$P(10, 3) = 10 \cdot 9 \cdot 8 = 720.$
Note that you start with 10 and multiply 3 numbers.

A general formula, using the multiplication principle:

$P(n, k) = n \cdot (n - 1) \cdot (n - 2) \cdots (n - k + 1)$.

Note that there are $k$ consecutive numbers on the right hand side.
Permutations

Example In how many ways can you choose a President, secretary and treasurer for a club from 12 candidates, if each candidate is eligible for each position, but no candidate can hold 2 positions? Why are conditions 1, 2 and 3 satisfied here?

$$P(12, 3) = 12 \times 11 \times 10 = 1,320.$$  
Condition 1 is satisfied because we have a single set of 12 candidates for all 3 positions.  
Condition 2 is satisfied because no one can hold more than one position.  
Condition 3 is satisfied because being president is different than being treasurer or secretary.
Permutations

**Example** You have been asked to judge an art contest with 15 entries. In how many ways can you assign 1\(^{st}\), 2\(^{nd}\) and 3\(^{rd}\) place? (Express your answer as \(P(n, k)\) for some \(n\) and \(k\) and evaluate.)

\[P(15, 3) = 15 \cdot 14 \cdot 13 = 2,730.\]

**Example** Ten students are to be chosen from a class of 30 and lined up for a photograph. How many such photographs can be taken? (Express your answer as \(P(n, k)\) for some \(n\) and \(k\) and evaluate.)

\[P(30, 10) = 30 \cdot 29 \cdot 28 \cdot 27 \cdot 26 \cdot 25 \cdot 24 \cdot 23 \cdot 22 \cdot 21.\] Note 30 \(-\) 10 \(=\) 20 and we stopped at 21.

\[P(30, 10) = 109,027,350,432,000\]
Example In how many ways can you arrange 5 math books on a shelf?

\[ P(5, 5) = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120 \]

The number \( P(n, n) = n \cdot (n - 1) \cdot (n - 2) \cdot \ldots \cdot 1 \) is denoted by \( n! \) or “\( n \) factorial”.

\( n! \) counts the number of ways that \( n \) objects can be arranged in a row.

\( n! \) grows fast: \( 1! = 1, 2! = 2, 2! = 6, 4! = 24, 5! = 120, 6! = 720, 7! = 5,040, 8! = 40,320, 9! = 362,880, 10! = 3,628,800, \ldots \) \( 59! \approx 10^{80} \) (roughly the number of particles in the universe)
Factorials

We can rewrite our formula for $P(n, k)$ in terms of factorials:

$$P(n, k) = \frac{n!}{(n - k)!}.$$ 

**Example** (a) Evaluate 12!

(b) Evaluate $P(12, 5)$.

$$12! = P(12, 12) = 12 \cdot 11 \cdots 2 \cdot 1 = 479,001,600.$$ 

$$P(12, 5) = \frac{12!}{7!} = \frac{479,001,600}{5,040} = 95,040.$$
Factorials

**Example** In how many ways can 10 people be lined up for a photograph?

$10! = P(10, 10)$.

**Example** How many three letter words (including nonsense words) can you make from the letters of the English alphabet, if letters cannot be repeated? (Express your answer as $P(n, k)$ for some $n$ and $k$ and evaluate.)

$P(26, 3) = 15,600$. 
Permutations of objects with some alike

**Example** How many words can we make by rearranging the letters of the word BEER?

The set \( \{ B, E, E, R \} = \{ B, E, R \} \) but we really have 4 letters with which to work. So let us start with the set \( \{ B, R, E, E \} \). We arrange them in \( 4! = 24 \) ways:

<table>
<thead>
<tr>
<th>BREE</th>
<th>BEER</th>
<th>RBEER</th>
<th>REBEE</th>
<th>REERB</th>
<th>EEBRE</th>
<th>EBREE</th>
<th>EBERE</th>
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If we can’t tell the difference between \( E \) and \( E \) (they are both just \( E \)), then the words group into *pairs*, e.g., \( EEBR \) and \( EEBR \) group together — both are the word EEBR.

Thus the number of different words we can form by rearranging the letters must be

\[
4!/2 = \frac{4!}{2!}
\]

Note that \( 2! \) counts the number of ways we can permute the two E’s in any given arrangement.
Permutations of objects with some alike

In general the number of permutations of $n$ objects with $r$ of the objects identical is

\[
\frac{n!}{r!}
\]

Note that $\frac{n!}{r!} = P(n, n - r)$.
Permutations of objects with some alike

**Example** How many words (including nonsense words) can be made from rearrangements of the word ALPACA?

\[
\frac{6!}{3!} = \frac{720}{6} = 120. \text{ There are 6 letters in ALPACA and one of them, 'A' is repeated 3 times.}
\]

**Example** How many words can be made from rearrangements of the word BANANA?

\[
\{B, A, N, A, N, A\} = \{A, B, N\}.
\]

The 'A' is repeated 3 times.
The 'N' is repeated 2 times.
The 'B' is repeated once.

Hence the answer is \[
\frac{6!}{1! \cdot 2! \cdot 3!} = 60.
\]
Permutations of objects with some alike

Suppose given a collection of $n$ objects containing $k$ subsets of objects in which the objects in each subset are identical and objects in different subsets are not identical. Then the number of different permutations of all $n$ objects is

$$\frac{n!}{r_1! \cdot r_2! \cdot \ldots \cdot r_k!}$$

where $r_1$ is the number of objects in the first subset, $r_2$ is the number of objects in the second subset, etc.

Note that if you make the collection of objects into a set, the set has $k$ elements in it.

Note that for a subset of size 1, we have $1! = 1$, so this formula is a generalization of the previous one.
Example How many words can be made from rearrangements of the letters of the word BOOKKEEPER?

\[
\frac{10!}{1! \cdot 3! \cdot 2! \cdot 2! \cdot 1! \cdot 1!} = 151,200.
\]

There are 10 letters in BOOKKEEPER. In alphabetical order, B ↔ 1, E ↔ 3, K ↔ 2, O ↔ 2, P ↔ 1, R ↔ 1.

Note that the total number of letters is the sum of the multiplicities of the distinct letters: 10 = 1+3+2+2+1+1.
Taxi cab geometry

In how many ways can a taxi drive from A to B, going the least possible number of blocks (nine)?

Two possible routes — SSSSEEEEEEEE in red and ESSEEESES in blue — are shown.
To go using the least number of blocks, the cab must always go South (S) or East (E), and in total must use 4 S’s and 5 E’s. Any rearrangement of SSSSEEEEE gives a valid route, and there are

$$\frac{9!}{4!5!}$$

such rearrangements.
Taxi cab geometry

**Example**  A streetmap of Mathville is given below. You arrive at the Airport at A and wish to take a taxi to Pascal’s house at P. The taxi driver, being an honest sort, will take a route from A to P with no backtracking, always traveling south or east.
(a) How many such routes are possible from A to P?

You need to go 4 blocks south and 5 blocks east for a total of 9 blocks so the number of routes is

\[
\frac{9!}{4! \cdot 5!} = \frac{9 \cdot 8 \cdot 7 \cdot 6}{4 \cdot 3 \cdot 2 \cdot 1} = 9 \cdot 2 \cdot 7 = 126.
\]
(b) If you insist on stopping off at the Combinatorium at C, how many routes can the taxi driver take from A to P?

This is really two taxicab problems combined with the Multiplication Principle. The answer, in words, is ‘the number of paths from A to C’ times ‘the number of paths from C to P’. The first is \( \frac{4!}{2! \cdot 2!} = 6 \) and the second is \( \frac{5!}{2! \cdot 3!} = 10 \) so the answer is \( 6 \cdot 10 = 60 \).
(c) If wish to stop off at both the combinatorium at C and the Vennitarium at V, how many routes can your taxi driver take?

This is three taxicab problems. The answer, in words, is ‘the number of paths from A to C’ times ‘the number of paths from C to V’ times ‘the number of paths from V to P. The first is \( \frac{4!}{2! \cdot 2!} = 6 \), the second is \( \frac{3!}{1! \cdot 2!} = 3 \) and the third is \( \frac{2!}{1! \cdot 1!} = 2 \) so the answer is \( 6 \cdot 3 \cdot 2 = 36 \).
(d) If you wish to stop off at either C or V (at least one), how many routes can the taxi driver take?

This problem involves both taxis and the Inclusion-Exclusion Principle. Suppose \( C \) denote the set of all paths from A to P that go through C and that \( V \) denotes the set of all paths from A to P that go through V.

The number we want is \( n(C \cup V) \) since \( C \cup V \) is the set of all paths which go through C or V.
Suppose $C$ denotes the set of all paths from A to P that go through C and that $V$ denotes the set of all paths from A to P that go through V.

The number we want is $n(C \cup V)$ since $C \cup V$ is the set of all paths which go through C or V.

We have already computed $n(C) = 60$. For $n(V)$ we have

$$n(V) = \frac{7!}{3! \cdot 4!} \cdot \frac{2!}{1! \cdot 1!} = \frac{7 \cdot 6 \cdot 5}{6} \cdot 2 = 70.$$  

We still need $n(C \cap V)$ but $C \cap V$ is the set of all paths which go through both C and V and we have already computed this: $n(C \cap V) = 36$.

Hence

$$n(C \cup V) = 60 + 70 - 36 = 94.$$

We can conclude that there are 94 ways to get from A to P via C or V.
Example Christine, on her morning run, wants to get from point A to point B.

(a) How many routes with no backtracking can she take?
(b) How many of those routes go through the point D?
(c) If Christine wants to avoid the Doberman at D, how many routes can she take?
Taxi cab geometry

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(b) How many of those routes go through the point D?
(c) If Christine wants to avoid the Doberman at D, how many routes can she take?

(a) \( \frac{(5 + 7)!}{5! \cdot 7!} \)

(b) \( \frac{(3 + 4)!}{3! \cdot 4!} \cdot \frac{(2 + 3)!}{2! \cdot 3!} \)

(c) \( \frac{(5 + 7)!}{5! \cdot 7!} - \left( \frac{(3 + 4)!}{3! \cdot 4!} \cdot \frac{(2 + 3)!}{2! \cdot 3!} \right) \)