# Linear Inequalities in Two Variables

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This is an example of a *linear inequality*.

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A line which runs through the point $(0, 0)$ has an equation of the form

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Minor technical issue: if $a = b = 0$ then either

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From plane geometry you know that the intersection of two lines is either the empty set (the lines are parallel), or the line (the lines are equal) or a single point.

You can find this single point (if it exists) with a little algebra.

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The point $(0, 0)$ does not satisfy the equation $2x + 3y = 6$, since $2(0) + 3(0) \neq 6$. Hence the point $(0, 0)$ is not on the graph of the equation $2x + 3y = 6$, and is not on the line.

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We can draw the graph of a line if we know the location of any 2 points on the line. The $x$- and $y$-intercepts (the places where the line hits the $x$-axis and the $y$-axis) are usually the easiest points to find.

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Given these two points (or any two distrint points) we can draw the line by joining the points with a straight edge and extending.

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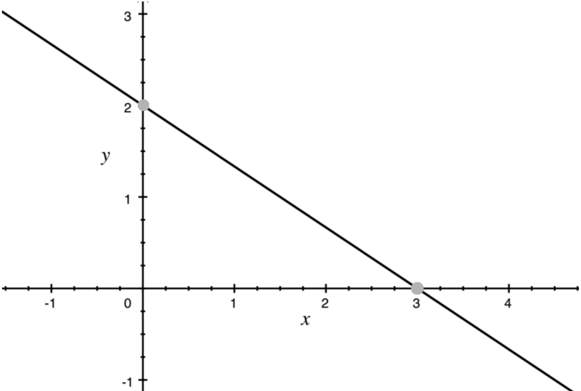

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- The graph of an equation of the form $y = d$ is a horizontal line which cuts the $y$-axis at $d$.
- The graph of an equation of the form $ax + by = 0$ cuts both axes at the point $(0, 0)$, so one needs to pick another value of $x$ (or $y$), find the corresponding value of $y$ (or $x$) with some algebra, and plot the corresponding point.

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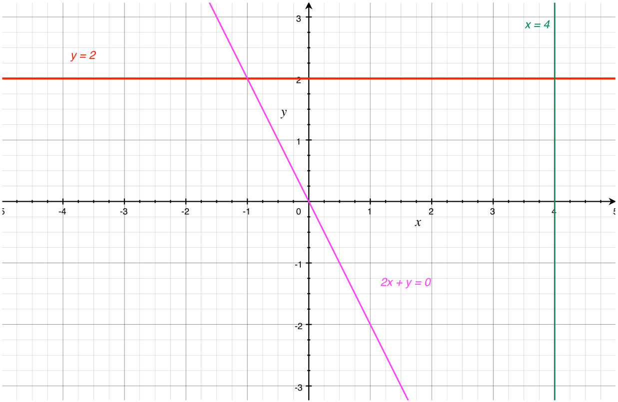

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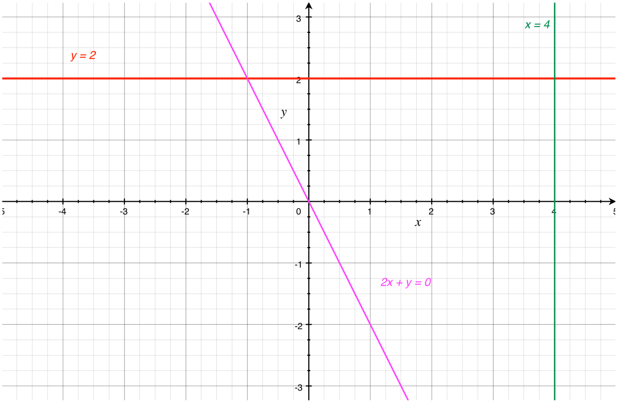


To draw $2x + y = 0$ note $(0, 0)$ is one point. Pick any non-zero value for $x$ and solve for $y$; if $x = -1$, $y = 2$.

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**Solution**: Here is one way to get the equation.

1. Find $a$ and $b$ by looking at the difference of the two points: $(x_1, y_1) - (x_0, y_0) = \big(x_1 - x_0, y_1 - y_0\big) = (b, -a)$.
2. Find $c$ by solving $c = ax_0 + by_0$.
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- Is the point $(x_0, y_0)$ on this line? YES, because $c = ax_0 + by_0$ (from Step 2)
- Is the point $(x_1, y_1)$ on this line? YES, because $c = ax_1 + by_1$ is the same as $ax_0 + by_0 = ax_1 + by_1$ (from Step 2), which is the same as $a(x_1 - x_0) + b(y_1 - y_0) = 0$, which is the same as $ab + b(-a) = 0$ (by Step 1), which is a correct equation.

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**Example**: Find the equation of the line through $(1, 0)$ and $(5, 0)$.

**Solution**:

1. $(5, 0) - (1, 0) = (4, 0)$ so $a = 0, b = 4$
2. $0x_0 + 4y_0 = 0 \cdot 1 + 4 \cdot 0 = 0$ so $c = 0$
3. The equation of the line is $\boxed{4y = 0.}$ (or $\boxed{y = 0.}$)

# Linear Inequalities in Two Variables

To solve some optimization problem, specifically *linear programming* problems, we must deal with **linear inequalities** of the form

$$\begin{aligned} ax + by &\geqslant c \\ ax + by &\leqslant c \\ ax + by &> c \\ ax + by &< c, \end{aligned}$$

where $a$, $b$ and $c$ are given numbers. Constraints on the values of $x$ and $y$ that we can choose to solve our problem, will be described by such inequalities.

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**Example**: Michael is taking a exam to become a volunteer firefighter. The exam has 10 essay questions and 50 short questions. Michael has 90 minutes to take the exam and knows he is not expected to answer every question. An essay question takes 10 minutes to answer and a short question takes 2 minutes. Let $x$ denote the number of short questions that Michael will attempt and let $y$ denote the number of essay questions that Michael will attempt. What linear inequalities describes the constraints on Michael's time given above?

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There is a time constraint: $\boxed{2x + 10y \leqslant 90}$.

Additionally, since Michael can't answer a negative number of questions, there are constraints $\boxed{x \geqslant 0}$ and $\boxed{y \geqslant 0}$. Furthermore, since he can not answer more questions than there are, so $\boxed{x \leqslant 50}$ and $\boxed{y \leqslant 10}$.

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Notice that any point $(x_1, y_1)$ satisfies **exactly one** of $ax + by > c$, $ax + by < c$ or $ax + by = c$.

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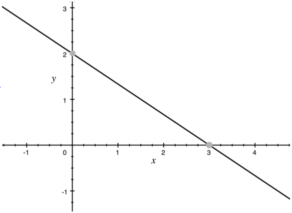

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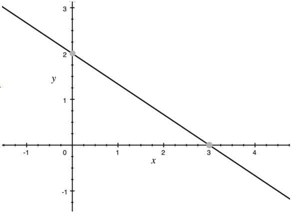


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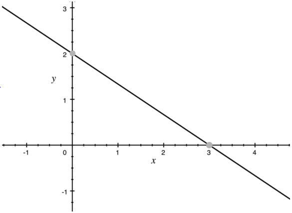


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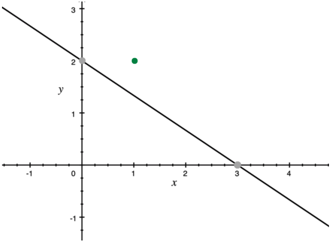

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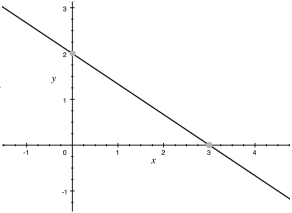

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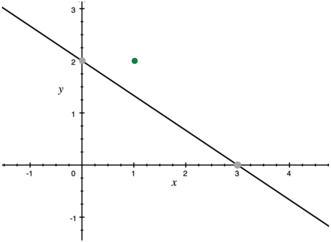

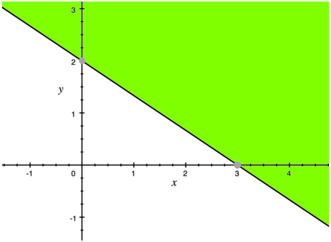

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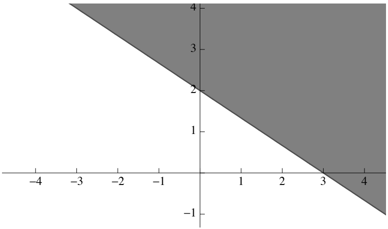


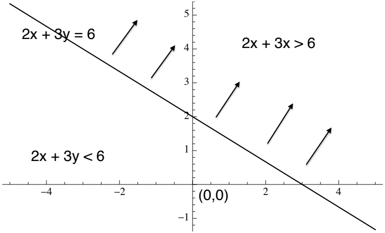

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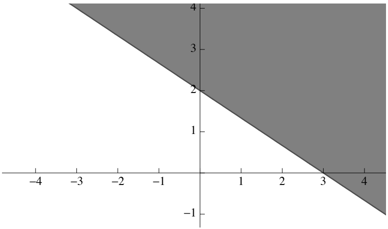


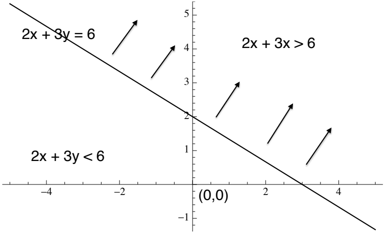


The plot with arrows will be more useful when we want to plot many inequalities simultaneously.

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**Graph of the inequality**
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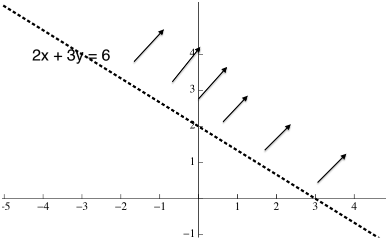

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Once you draw the line, it should be easy to pick out the upper/right and lower/left half-planes.

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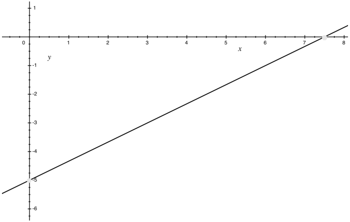
- Same technique works for $ax + by < c$, $ax + by \geq c$ and $ax + by \geq c$.
- If $ax + by = c$ doesn't pass through $(0, 0)$, $(0, 0)$ is an excellent choice of test point.

# Example: $2x - 3y \geqslant 15$

Here's the graph of $2x - 3y = 15$:



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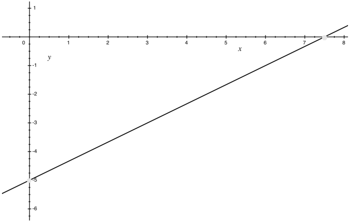
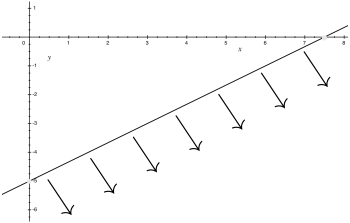



At $(0,0)$, $2x - 3y = 0 < 15$ which is less than 15. So we need to shade the side of the line opposite $(0,0)$:



# Example: $x - 3y \geqslant 0, \quad x > 2, \qquad y \leqslant 10$

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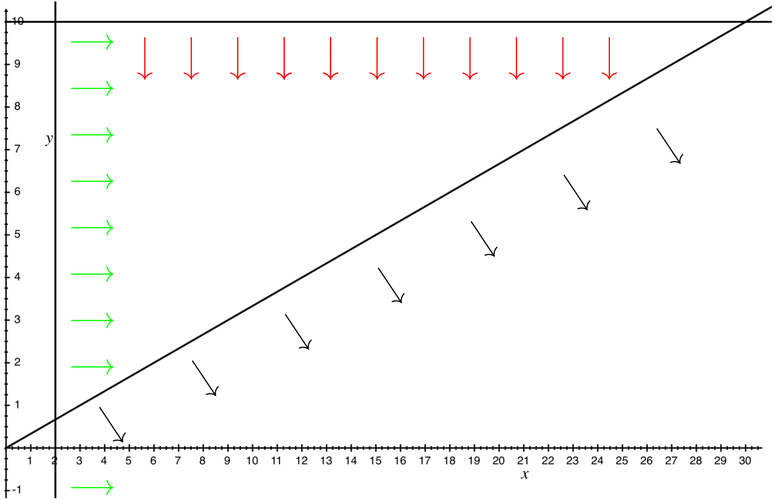

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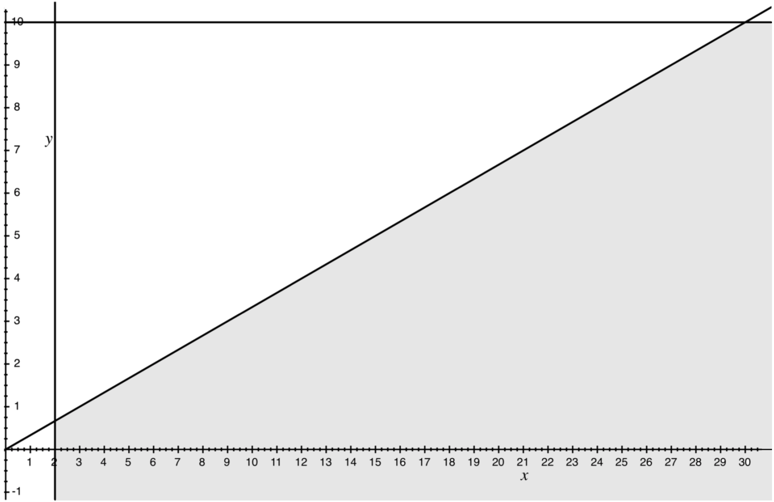

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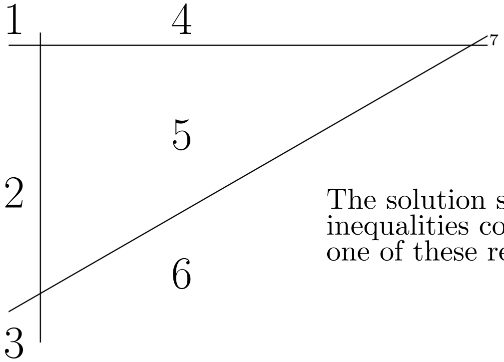
- 


The solution set of your inequalities consists of exactly one of these regions

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For the example $x - 3y \geqslant 0, x > 2, y \leqslant 10$, this process is illustrated on the next three slides.

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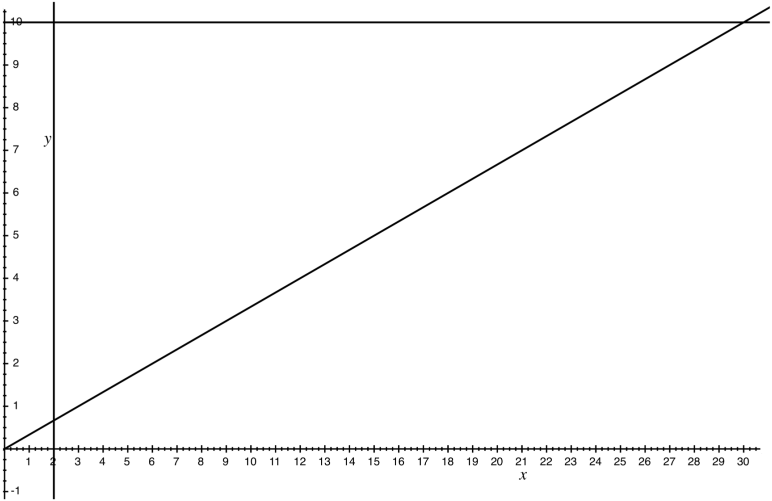

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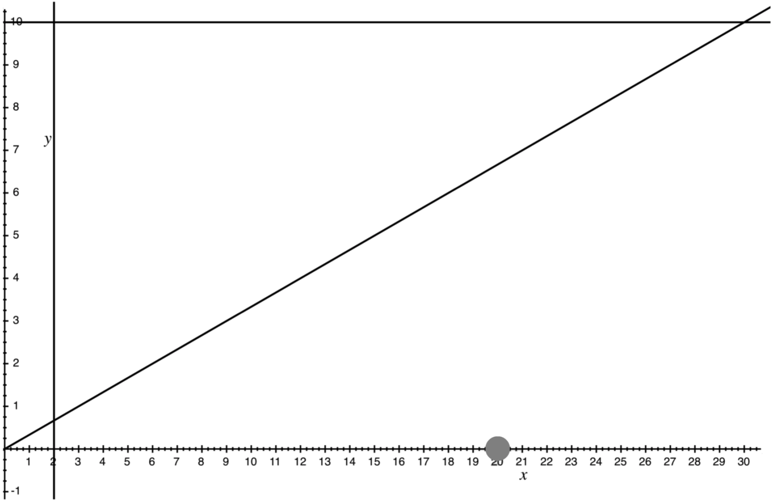

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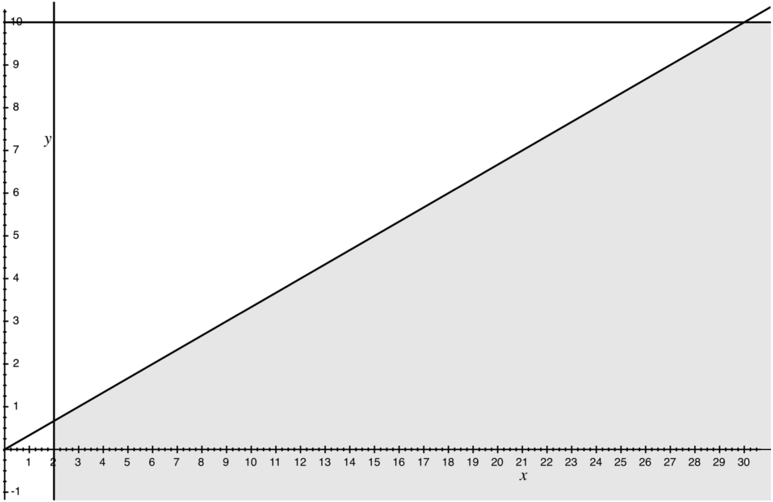

# Example

Returning to a previous example: Michael is taking a exam to become a volunteer firefighter. The exam has 10 essay questions and 50 short questions. Michael has 90 minutes to take the exam and knows he is not expected to answer every question. An essay question takes 10 minutes to answer and a short question takes 2 minutes. Let $x$ denote the number of short questions that Michael will attempt and let $y$ denote the number of essay questions that Michael will attempt.

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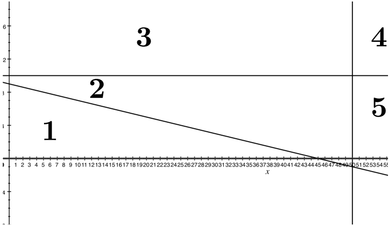
- Time constraint $\boxed{2x + 10y \leqslant 90}$.
- $\boxed{x \geqslant 0}$ and $\boxed{y \geqslant 0}$.
- $\boxed{x \leqslant 50}$ and $\boxed{y \leqslant 10}$.

# Example

Here are the lines $2x + 10y = 90$, $x = 0$, $y = 0$, $x = 50$, $y = 10$.

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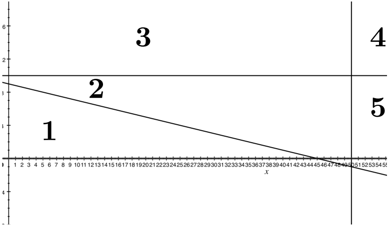


- We've only numbered regions in the *first quadrant* of the plane where $x \geq 0$, $y \geq 0$ since these two constraints must be satisfied. This saves a lot of labor!
- $x \leqslant 50$ and $y \leqslant 10$ are redundant — we can't have $2x + 10y \leqslant 90$ if $x > 50$ or $y > 10$. But you often can't tell if a constraint is redundant until you draw the graph, so it's best to err on the side of caution and graph too many constraints rather than graph too few.

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The solution: