The Monty Hall game

Game show host Monty Hall asks you to choose one of three doors. Behind one of the doors is a new Porsche. Behind the other two doors there are goats. Monty knows what is behind each door, you don't.

After you choose, Monty opens one of the remaining doors, revealing a goat. He then asks you whether you want to switch your choice to the other unopend door, or stick with your original choice. Once you make your choice, Monty opens the chosen door and you win whatever is behind the door.

If you want to make your chance of winning the car as high as possible, is it

1. better to stick
2. better to switch
3. or does your second choice make no difference?

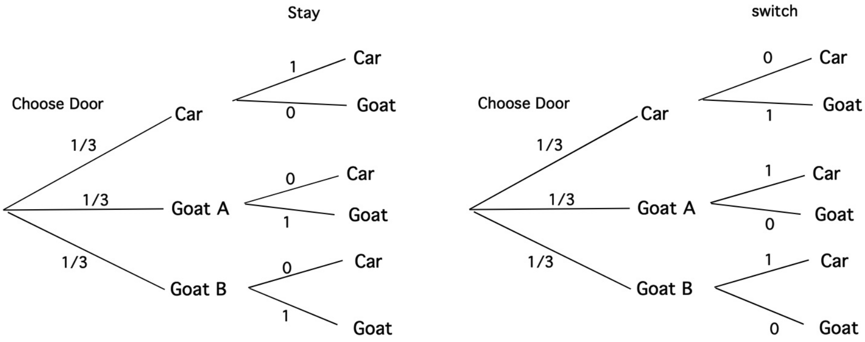
Strategies in games of chance

We can use probability as a decision making tool here and choose the strategy that maximizes our probability of success. (This decision making tool makes most sense if a player is playing the game repeatedly or if they are part of a pool of players playing the game repeatedly).

On the next page is a tree diagram labelled with probabilities associated to each strategy. The player's winnings are shown at the end of each path.

For the interesting social history of the Monty Hall problem, see for example `https://en.wikipedia.org/wiki/Monty_Hall_problem`.

Strategies in games of chance



We see that the probability of winning the car is 2/3 if the player switches and just 1/3 if the player stays with their original choice. Thus using probability as a decision making tool, we would choose to switch doors.

Strategies in games of chance

Example: Roulette

In a game of American roulette where you bet \$1 on red, the probability distribution for your earnings X is given by:

X	**P**(X)
1	18/38
-1	20/38

The expected earnings for this game are $\mathbf{E}(X) = -2/38$.

Because the expected earnings are negative, you see that your earnings would average out as a *loss* in the long run and you would choose not to play this game repeatedly.

Strategies in two-person zero-sum games

In general when choosing a strategy, we aim for a higher probability of winning or a higher expected pay-off. However when choosing a strategy for game theory, our opponent is no longer chance. We have an opponent who

1. wishes to maximize their own gain (which for a zero-sum game equates to minimizing our gain),
2. has full knowledge of the consequences of each strategy from the pay-off matrix, and assumes that we will play to our own advantage,
3. can use statistics to notice patterns in our play if we are repeatedly playing the game and can anticipate the logical steps we might take to change our strategy and improve our pay-off.

The situation is thus quite dynamic: we need to take into account how our opponent is likely to play and how they will respond to any particular strategy we choose for repeated play.

Strictly Determined Games

We'll start with a special type of game called a **strictly determined game** where the best thing for both players is to choose a single strategy/option (a **fixed** or a **pure strategy**) and stick with it on every play.

Example: The coin matching game from last time. Roger and Colleen play a game. Each one has a coin. They will both show a side of their coin simultaneously. If both show heads, no money will be exchanged. If Roger shows heads and Colleen shows tails then Colleen will give Roger \$1. If Roger shows tails and Colleen shows heads, then Roger will pay Colleen \$1. If both show tails, then Colleen will give Roger \$2.

The pay-off matrix for Roger in shown on the next slide.

Strictly Determined Games

		Colleen	
R		H	T
o	H	0	1
g.	T	-1	2

Should Roger play tails, since that offers the possibility of the maximum payoff, 2? No!

- If Colleen learns this (e.g. through repeated play), she will start playing heads to win \$1.
- In fact, Colleen clearly will *always* play heads: her worst payoff playing heads (0) is better than her best payoff playing tails (lose \$1).
- Knowing Colleen is certain to play Heads, Roger should play Heads, (otherwise he will lose a dollar) and his payoff is zero.

Strictly Determined Games

Colleen

		H	T
Rog.	H	0	1
	T	-1	2

When both players play Heads, neither player can gain by changing their strategy.

- If Colleen switches to tails, she goes from breaking even to losing a dollar.
- If Roger switches to tails, he goes from breaking even to losing a dollar.

The combination Rog: H and Coll: H is a point of equilibrium. We say that the matrix has a **saddle point** at Row 1, Col 1. The entry of the matrix at this point (0 payoff for R for this combination of strategies) is called the **value** of the game.

Recap of assumptions

In general when I am contemplating what my best fixed strategy might be, I need to keep three things in mind:

- I wish to maximize my payoff;
- both I and my opponent have full knowledge of the payoff matrix; and
- wishes to maximize her own payoff.

Note: in a zero-sum game, the column player maximizes their payoff by minimizing the row player's payoff.

Finding fixed strategies

Suppose we have two players, R and C, where R has m strategies $r_1, r_2, \ldots, r_m$ and C has n strategies $c_1, c_2, \ldots, c_n$, with payoff matrix

$$
\begin{array}{cc|cccc}
 & & & \mathbf{C} & & \\
 & & c_1 & c_2 & \ldots & c_n \\
\hline
 & r_1 & a_{11} & a_{12} & \ldots & a_{1n} \\
 & r_2 & a_{21} & a_{22} & \ldots & a_{2n} \\
\mathbf{R} & \vdots & \vdots & \vdots & \vdots & \vdots \\
 & r_m & a_{m1} & a_{m2} & \ldots & a_{mn}
\end{array}
$$

If R is going to play the same fixed strategy repeatedly, we can assume that C will minimize R's payoff by choosing the strategy (column) which corresponds to the minimum payoff (for R) in the row. Thus for each row of the payoff matrix, R can assume that their payoff will be the *minimum* entry in that row if they choose that strategy.

The optimum fixed (pure) strategy for R

For each row of the pay-off matrix (R's pay-off matrix), find the least element, and note it in a column to the right of the payoff matrix.

C

		c_1	c_2	$\dots$	c_n	Min
	r_1	a_{11}	a_{12}	$\dots$	a_{1n}	min of row 1
	r_2	a_{21}	a_{22}	$\dots$	a_{2n}	min of row 2
R	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	r_m	a_{m1}	a_{m2}	$\dots$	a_{mn}	min of row m

Locate the row for which this element is largest (the row corresponding to the *maximum* of the numbers in the right-hand column above). The option corresponding to this row is the best fixed strategy for R.

The optimum fixed (pure) strategy for R

Example: Suppose Cat and Rat play a game where the pay-off matrix for Rat is given by:

	C_1	C_2	C_3
R_1	1	-1	0
R_2	6	-3	1
R_3	2	2	1

Find the optimal fixed strategy for R.

	C_1	C_2	C_3	min
R_1	1	-1	0	-1
R_2	6	-3	1	-3
R_3	2	2	1	1

The largest number in the min column is 1, and hence the optimal fixed strategy for R is R_3.

The optimum fixed (pure) strategy for C

Whatever strategy (column) C settles on, row player will choose the row which gives the maximum payoff for R. So to find an optimal fixed strategy C needs to find the largest element of each column of the pay-off matrix (R's pay-off matrix), and record this is a new row along the bottom of the array,

C

R		c_1	c_2	$\ldots$	c_n
	r_1	a_{11}	a_{12}	$\ldots$	a_{1n}
	r_2	a_{21}	a_{22}	$\ldots$	a_{2n}
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	r_m	a_{m1}	a_{m2}	$\ldots$	a_{mn}
	Max	max. of Col 1	max. of Col 2	$\ldots$	max. of Col n

then locate the column for which this element is smallest (the column corresponding to the *minimum* of the numbers in the bottom row above). The option corresponding to this column is the best fixed strategy for C.

The optimum fixed (pure) strategy for C

Example: Find the optimal fixed strategy for Cat in the previous game.

	C_1	C_2	C_3
R_1	1	-1	0
R_2	6	-3	1
R_3	2	2	1

	C_1	C_2	C_3
R_1	1	-1	0
R_2	6	-3	1
R_3	2	2	1
max	6	2	1

The smallest number in the max column is 1, and hence the optimal fixed strategy for C is C_3.

Saddle points

If the largest of the row minima and the smallest of the column maxima occur at the same entry of the payoff matrix, then we say that the matrix has a **saddle point** at that location.

In this case, we say the game is **strictly determined** and the value of the matrix entry at the saddle point is called the **value of the game**.

The best strategy for both players (the one which gives the maximum expected payoff if the game is played repeatedly and both players play intelligently) is to play the fixed strategies corresponding to the saddle point. These two strategies are called a **solution** to the game.
You can easily recognize a saddle point: *it is a point that is a minimum in its row and a maximum in its column.*

Multiple saddle points

There may be more than one saddle point in a payoff matrix, in which case, there is more than one possible solution. However, in this case the value of the entry at each possible solution is the same and thus the value of the game is unique. The game is still considered to be strictly determined with a unique value, however the players may have more than one option for an optimal strategy.

Example:

	C1	C2	C3	C4	C5
R1	6	1	-2	-1	5
R2	5	2	3	2	4
R3	7	0	6	1	-5
R4	3	2	5	2	5
R5	4	1	7	0	10

Multiple saddle points

	C1	C2	C3	C4	C5	min		
R1	6	1	-2	-1	5	-2		
R2	5	2	3	2	4	2		
R3	7	0	6	1	-5	-5		
R4	3	2	5	2	5	2		
R5	4	1	7	0	10	0		
max	7	2	7	2	10		2	min of max's
						2		
						max of min's		

The saddle points of this game are R2 C2, R2 C4, R4 C2 and R4 C4, and all lead to the same value of the game, 2.

Rat and Cat example

	C_1	C_2	C_3
R_1	1	-1	0
R_2	6	-3	1
R_3	2	2	1

(a) Is this game between Cat and Rat a strictly determined game?

This game is strictly determined since the maximum row minima and the minimum column maximum both occur at position 3 3. Said another way, the game has a saddle point which occurs at position 3 3. The solution of the game is for Rat to play R3 and Cat to play C3.

(b) What is the value of the game?

The value of the game is 1.

An example from Imperial Russia

Catherine (C) and Rasputin (R) play this game:

	$C1$	$C2$	$C3$
$R1$	0	1	4
$R2$	5	-1	3
$R3$	3	2	5

(a) Is this a strictly determined game?

	$C1$	$C2$	$C3$	min
$R1$	0	1	4	0
$R2$	5	-1	3	-1
$R3$	3	2	5	2
max	5	2	5	

This game is strictly determined since the maximum row minima and the minimum column maximum both occur at position 3 2, or $(r, c) = (3, 2)$. The value of the game is 2.

An example from Imperial Russia

	$C1$	$C2$	$C3$	min
$R1$	0	1	4	0
$R2$	5	-1	3	-1
$R3$	3	2	5	2
max	5	2	5	

(b) Sometimes in reality players do not play optimally: If Rasputin always plays R1, which column should Catherine play in order to maximize her gain?

If Rasputin can be counted on to play R1, Catherine should counter by playing C1, decreasing her payout from 2 to 0.

One more example

Romeo (R) and Collette (C) play a zero-sum game for which the payoff matrix for Romeo is given by:

	$C1$	$C2$	$C3$	$C4$	$C5$
$R1$	1	5	9	1	4
$R2$	-3	-1	-3	-2	7
$R3$	-2	-3	1	-9	8
$R4$	1	2	2	1	14

(a) Find all saddle points for this matrix.

One more example

	$C1$	$C2$	$C3$	$C4$	$C5$	min
$R1$	1	5	9	1	4	1
$R2$	-3	-1	-3	-2	7	-3
$R3$	-2	-3	1	-9	8	-9
$R4$	1	2	2	1	14	1
max	1	5	9	1	14	

There are four saddle points: at $(1, 1)$, $(1, 4)$, $(4, 1)$ and $(4, 4)$.
(b) What is the value of the game?
At each saddle point the payoff is 1. The value is 1.
(c) What are the possible solutions to the game?
Each player has 2 optimal strategies. Collette can play either C1 or C4. No matter which strategy Collette plays, Romeo can play either R1 or R4, so there are 4 solutions to the game $(R1, C1)$, $(R1, C4)$, $(R4, C1)$ and $(R4, C4)$.

A word of caution

Not every game is strictly determined, in which case a fixed strategy for both players is not optimal. We will discuss mixed strategies and optimal strategies for these games in the next two lectures. Meanwhile, here are some examples.

Roger and Colleen's game

Is Roger and Colleen's coin matching game strictly determined?

Colleen

R		H	T
o	H	0	1
g.	T	-1	2

Colleen

R		H	T	min
o	H	0	1	0
g.	T	-1	2	-1
	max	0	2	

This game has a saddle point at $(1,1)$. Hence it is strictly determined and has a value of 0.

Coyote and Roadrunner's game

Is Coyote and Roadrunner's bombing game strictly determined?

		C. attacks	
		B	S
R.	B	80%	100%
places bomb	S	90%	50%

		C. attacks		
		B	S	*min*
R.	B	80%	100%	80%
places bomb	S	90%	50%	50%
max		90%	100%	

The minimum of the maxima is 90 and the maxima of the minima is 80 so there is no saddle point.

Charlie and Ruth's Two-finger Morra

Charlie

		(1,2)	(1,3)	(2,3)	(2,4)
R	(1,2)	0	2	−3	0
u	(1,3)	−2	0	0	3
t	(2,3)	3	0	0	−4
h	(2,4)	0	−3	4	0

Charlie

		(1,2)	(1,3)	(2,3)	(2,4)	*min*
R	(1,2)	0	2	−3	0	−3
u	(1,3)	−2	0	0	3	−2
t	(2,3)	3	0	0	−4	−4
h	(2,4)	0	−3	4	0	−3
max		3	2	4	3	

The minimum of the maxima is 2 and the maxima of the minima is -2 so there is no saddle point.

A few last examples

For the following payoff matrices, find the saddle points, values and solutions if they exist.

$$\begin{bmatrix} -1 & -3 & 3 \\ 5 & 0 & 2 \\ 6 & -1 & 1 \end{bmatrix} \qquad \begin{bmatrix} -1 & 4 & 3 \\ 5 & 2 & 2 \\ 6 & -1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 1 \\ 5 & 2 \\ 6 & -2 \end{bmatrix} \qquad \begin{bmatrix} -1 & 1 \\ -5 & 2 \\ -1 & 2 \end{bmatrix}$$

Solutions

$$\begin{array}{rrrcr} -1 & -3 & 3 & & -3 \\ 5 & 0 & 2 & & 0 \\ 6 & -1 & 1 & & -1 \\ \\ 6 & 0 & 3 & & \end{array}$$

Saddle point at $(2, 2)$ with game value 0.

$$\begin{array}{rrrcr} -1 & 4 & 3 & & -1 \\ 5 & 2 & 2 & & 2 \\ 6 & -1 & 1 & & -1 \\ \\ 6 & 4 & 3 & & \end{array}$$

The minimum of the maxima is 3 and the maxima of the minima is 2 so there is no saddle point.

Solutions

$$\begin{array}{rrr} -1 & 1 & -1 \\ 5 & 2 & 2 \\ 6 & -2 & -2 \\ \\ 6 & 2 & \end{array}$$

Saddle point at $(2, 2)$ with game value 2.

$$\begin{array}{rrr} -1 & 1 & -1 \\ -5 & 2 & -5 \\ -1 & 2 & -1 \\ \\ -1 & 2 & \end{array}$$

The minimum of the maxima is -1 and the maxima of the minima is -1 so there is a saddle point(s). Saddle points at $(1, 1)$ and $(3, 1)$ with game value -1.