1. Which of the following expressions gives the partial fraction decomposition of the function
\[ f(x) = \frac{3x^2 + 2x + 1}{(x - 1)(x^2 - 1)(x^2 + 1)} \]?

**Solution:** Notice that \((x^2 - 1)\) is not an irreducible factor. If we write the denominator in terms of irreducible factors we get
\[ f(x) = \frac{3x^2 + 2x + 1}{(x - 1)^2(x + 1)(x^2 + 1)} \]
since \((x^2 - 1) = (x - 1)(x + 1)\). Thus we see that the final answer should be
\[
\frac{A}{x - 1} + \frac{B}{(x - 1)^2} + \frac{C}{x + 1} + \frac{Dx + E}{x^2 + 1}
\]

2. Use the Trapezoidal rule with step size \(\Delta x = 1\) to approximate the integral \(\int_0^4 f(x)dx\) where a table of values for the function \(f(x)\) is given below.

<table>
<thead>
<tr>
<th>(x)</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>(f(x))</td>
<td>2</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>5</td>
</tr>
</tbody>
</table>

**Solution:** Using the formula for the trapezoidal rule with \(\Delta x=1\) we see that
\[
\int_0^4 f(x)dx \approx \frac{\Delta x}{2} (f(0) + 2f(1) + 2f(2) + 2f(3) + f(4)) = \frac{1}{2}(2 + 2 + 4 + 6 + 5) = \frac{19}{2} = 9.5
\]

3. Evaluate the integral \(\int_2^\infty xe^{-x} \, dx\).

**Solution:** First we find the indefinite integral using integration by parts: Let \(u = x\) and \(dv = e^{-x}dx\) so that \(du = dx\) and \(v = -e^{-x}\). So we have that
\[
\int xe^{-x} \, dx = -xe^{-x} - \int -e^{-x} \, dx = -xe^{-x} - e^{-x} + C
\]
Then we see that
\[
\int_2^\infty xe^{-x} \, dx = \lim_{b \to \infty} \int_2^b xe^{-x} \, dx = \lim_{b \to \infty} \left(-xe^{-x} - e^{-x}\right) \bigg|_2^b = \lim_{b \to \infty} \left((be^{-b} - e^{-b}) - (-2e^{-2} - e^{-2})\right) = 0 - (-3e^{-2}) = \frac{3}{e^2}
\]
4. Compute the integral
\[ \int_{-3}^{3} \frac{1}{(x+2)^3} \, dx. \]

**Solution:** We have to be careful at the point where the function does not exist, namely \( x = -2 \). So we see that
\[ \int_{-3}^{3} \frac{1}{(x+2)^3} \, dx = \int_{-3}^{-2} \frac{1}{(x+2)^3} \, dx + \int_{-2}^{3} \frac{1}{(x+2)^3} \, dx. \]
We work first on the part \( \int_{-2}^{3} \frac{1}{(x+2)^3} \, dx \). We will solve this using \( u \)-substitution. If we let \( u = x + 2 \) (so \( du = dx \)), then the bounds change from \( x = -2 \) to \( u = 0 \) and \( x = 3 \) to \( u = 5 \). Making the substitution we see that
\[ \int_{-2}^{3} \frac{1}{(x+2)^3} \, dx = \int_{0}^{5} \frac{1}{u^3} \, du = \lim_{b \to 0} \left( \int_{b}^{5} u^{-3} \, du \right) \]
\[ = \lim_{b \to 0} \left( \frac{-u^{-2}}{2} \right)_{b}^{5} = \lim_{b \to 0} \left( \frac{-5^{-2}}{2} + \frac{b^{-2}}{2} \right) = \lim_{b \to 0} \left( \frac{-1}{50} + \frac{1}{2b^2} \right) = \infty \]
So the integral is **divergent**.

5. Compute the integral
\[ \int_{0}^{\frac{\pi}{2}} \cos (\cos (x)) \sin (x) \, dx. \]

**Solution:** We solve this by \( u \)-substitution. Let \( u = \cos (x) \) (so \( du = -\sin (x) \, dx \)). Then the bounds of integration change from \( x = \frac{\pi}{2} \) to \( u = 0 \) and from \( x = 0 \) to \( u = 1 \). Making the substitutions we get
\[ \int_{0}^{\frac{\pi}{2}} \cos (\cos (x)) \sin (x) \, dx = \int_{1}^{0} -\cos (u) \, du \]
\[ = -\sin (u) \bigg|_{1}^{0} = -\sin (0) - (-\sin (1)) = \sin (1) \]

6. Which of the following is an expression of the area of the surface formed by rotating the curve \( y = \sin x \) between \( x = 0 \) and \( x = \frac{\pi}{2} \) about the \( x \)-axis?

**Solution:** The formula is given by
\[ \int_{a}^{b} 2\pi y \sqrt{1 + \left( \frac{dy}{dx} \right)^2} \, dx \]
where in our situation \( a = 0 \), \( b = \frac{\pi}{2} \), \( y = \sin (x) \) and so \( \frac{du}{dx} = \cos (x) \). Plugging all in and pulling the \( 2\pi \) out we get:
\[ 2\pi \int_{0}^{\frac{\pi}{2}} \sin (x) \sqrt{1 + \cos^2 (x)} \, dx \]
7. Find the centroid of the region bounded by \( y = e^x, y = 0, x = 0 \) and \( x = 1 \).

**Solution:** First we note that the area of the region \( A \) is given by

\[
A = \int_0^1 e^x \, dx = e^x \bigg|_0^1 = e^1 - e^0 = e - 1
\]

Now, we find the centroid by finding \( \bar{x} \) and \( \bar{y} \):

\[
\bar{x} = \frac{1}{A} \int_0^1 x e^x \, dx, \quad \bar{y} = \frac{1}{A} \int_0^1 \frac{1}{2} (e^x)^2 \, dx
\]

For \( \bar{x} \), we solve the integral using integration by parts with \( u = x \) and \( dv = e^x \, dx \) so that \( du = dx \) and \( v = e^x \). Then we get that \( \int x e^x \, dx = x e^x - \int e^x \, dx = x e^x - e^x + C \).

Using this we get

\[
\bar{x} = \frac{1}{e - 1} \left[ x e^x - e^x \right]_0^1 = \frac{1}{e - 1} \left( (e - e) - (0 - 1) \right) = \frac{1}{e - 1}
\]

For \( \bar{y} \) we note that \((e^x)^2 = e^{2x}\). Then we use \( u \)-substitution with \( u = 2x \) so that \( du = 2dx \) and the bounds change from \( x = 0 \) to \( u = 0 \) and from \( x = 1 \) to \( u = 2 \).

Making the substitution we get

\[
\bar{y} = \frac{1}{e - 1} \left( e - 1 \right) \left( e^u \right)_0^2 = \frac{1}{e - 1} \left( e^2 - 1 \right) = \frac{e + 1}{4}.
\]

Thus the centroid lies at the coordinates \( \left( \frac{1}{e - 1}, \frac{e + 1}{4} \right) \).

8. Use Euler’s method with step size 0.5 to estimate \( y(2) \) where \( y(x) \) is the solution to the initial value problem

\[
y' = (x - 1)(y - x), \quad y(1) = 2.
\]

**Solution:** This will require two steps in Euler’s method. For step one, we know that \( x_0 = 1 \) and \( y_0 = 2 \). Additionally, we know that \( h = 0.5 \). We also know that \( x_1 = 1.5 \) and \( x_2 = 2 \) so we can stop at step 2.

\[
y_1 = y_0 + h(x_0 - 1)(y_0 - x_0) = 2 + (.5)(1)(0) = 2
\]

\[
y_2 = y_1 + h(x_1 - 1)(y_1 - x_1) = 2 + (.5)(1.5 - 1)(2 - 1.5) = 2 + (.5)^3 = 2.125
\]

9. Compute the arc length of the curve \( y = \frac{2}{3} x^{\frac{3}{2}} \) from \( x = 0 \) to \( x = 3 \).

**Solution:** We see that \( \frac{dy}{dx} = x^{\frac{1}{2}} = \sqrt{x} \). Plugging into the formula for arc length we get that

\[
\text{arc length} = \int_0^3 \sqrt{1 + (\sqrt{x})^2} \, dx = \int_0^3 \sqrt{1 + x} \, dx = \frac{2}{3} \left( (x + 1)^{\frac{3}{2}} \right)_0^3
\]

\[
= \frac{2}{3} \left( 4^{\frac{3}{2}} - 1^{\frac{3}{2}} \right) = \frac{2}{3} (8 - 1) = \frac{14}{3}.
\]
10. Compute the integral
\[ \int \frac{x^2 + 2x}{x^2 - 1} \, dx. \]

**Solution:** First we do long division dividing \( x^2 - 1 \) into \( x^2 + 2x \). Doing this we get that
\[ \frac{x^2 + 2x}{x^2 - 1} = 1 + \frac{2x + 1}{x^2 - 1} \]
and
\[ \int \frac{x^2 + 2x}{x^2 - 1} \, dx = \int 1 \, dx + \int \frac{2x + 1}{x^2 - 1} \, dx \quad (1) \]
The first integral in (1) is straightforward: \( \int 1 \, dx = x + C \). The second integral is obtained using integration by partial fractions. By partial fractions we obtain:
\[ \frac{2x + 1}{x^2 - 1} = \frac{2x + 1}{(x - 1)(x + 1)} = \frac{A}{x + 1} + \frac{B}{x - 1} \]
So we have that
\[ 2x + 1 = A(x - 1) + B(x + 1) \]
Plugging in \( x = 1 \) gives \( 2B = 3 \) and plugging in \( x = -1 \) gives \( -2A = -1 \), so we see that \( A = \frac{1}{2} \) and \( B = \frac{3}{2} \). Using this decomposition gives
\[ \int \frac{2x + 1}{x^2 - 1} \, dx = \int \frac{1}{x + 1} \, dx + \int \frac{3}{x - 1} \, dx = \frac{1}{2} \ln |x + 1| + \frac{3}{2} \ln |x - 1| + C \]
Putting it all together, (1) becomes:
\[ \int \frac{x^2 + 2x}{x^2 - 1} \, dx = x + \frac{1}{2} \ln |x + 1| + \frac{3}{2} \ln |x - 1| + C \]

11. Evaluate the integral
\[ \int_0^1 (1 - \sqrt{x})^8 \, dx. \]

**Solution:** We do this with \( u \)-substitution. Let \( u = 1 - \sqrt{x} \) so that \( \sqrt{x} = 1 - u \) and hence \( x = (1 - u)^2 \). Using this, we see that \( dx = -2(1 - u)du \). Also, the bounds of integration go from \( x = 0 \) to \( u = 1 \) and from \( x = 1 \) to \( u = 0 \). Making the substitution gives:
\[ \int_0^1 (1 - \sqrt{x})^8 \, dx = \int_0^1 -2(1 - u)u^8 \, du = 2 \int_0^1 (u^8 - u^9) \, du = 2 \left( \frac{u^9}{9} - \frac{u^{10}}{10} \right) \bigg|_0^1 = 2 \left( \left( \frac{1}{9} - \frac{1}{10} \right) - 0 \right) = 2 \left( \frac{1}{90} \right) = \frac{1}{45}. \]
12. Find the solution to the initial value problem
\[(1 - x)y' - y^2 = 1, \quad y(2) = 1.\]

**Solution:** We can make this into a separable equation in the following way:
\[(1 - x)y' = y^2 + 1\]
Now, separate and integrate to find the solution:
\[\frac{1}{y^2 + 1} dy = \frac{1}{1 - x} dx\]
and so
\[\int \frac{1}{y^2 + 1} dy = \int \frac{1}{1 - x} dx\]
\[\tan^{-1}(y) = -\ln |x - 1| + C\]
To solve for $C$ we use the initial value $y(2) = 1$ giving us that $\tan^{-1}(1) = -\ln |2 - 1| + C$ which implies that $C = \tan^{-1}(1) = \frac{\pi}{4}$. Solving for $y$ we get
\[y = \tan \left(\frac{\pi}{4} - \ln(x - 1)\right)\]

13. Solve the initial value problem
\[y' = \frac{2x - y}{1 + x}, \quad y(1) = 2.\]

**Solution:** We first rewrite it as $y' = \frac{2x}{1 + x} - \frac{y}{1 + x}$ which allows us to rewrite as
\[y' + \frac{y}{x + 1} = \frac{2x}{x + 1}\]
Now, it is in standard form for a first-order linear differential equation with $P(x) = \frac{1}{x + 1}$ and $Q(x) = \frac{2x}{x + 1}$. We find the integrating factor (noting $\int P(x)dx = \int \frac{1}{x + 1} dx = \ln |x + 1|$):
\[I(x) = e^{\int P(x)dx} = e^{\ln |x + 1|} = x + 1.\]
So the final solution is given by
\[y(x) = \frac{1}{I(x)} \left( \int I(x)Q(x) \, dx \right) = \frac{1}{x + 1} \left( \int (x + 1) \left( \frac{2x}{x + 1} \right) \, dx \right)\]
\[= \frac{1}{x + 1} \int 2x \, dx = \frac{1}{x + 1} \left( x^2 + C \right)\]
Using the initial value $y(1) = 2$ tells us that $2 = \frac{1}{2}(1 + C)$ which means $C = 3$. So finally we have that
\[y(x) = \frac{x + 1}{x^2 + 3}\]