SOLUTIONS TO PRACTICE EXAM 3, MATH 10560

1. Calculate
\[ \lim_{n \to \infty} \frac{(\ln n)^2}{n} . \]

**Solution:** Use L’Hospital’s rule (twice):
\[
\lim_{x \to \infty} \frac{(\ln x)^2}{x} = \lim_{x \to \infty} \frac{2 \ln x}{x} = \lim_{x \to \infty} \frac{2}{x} = 0.
\]

2. Find
\[ \sum_{n=1}^{\infty} \frac{2^{2n}}{3 \cdot 5^{n-1}} . \]

**Solution:**
\[
\sum_{n=1}^{\infty} \frac{2^{2n}}{3 \cdot 5^{n-1}} = \frac{4^n}{3 \cdot 5^{n-1}} = \frac{4^n}{3} \sum_{n=1}^{\infty} \left( \frac{5}{4} \right)^{n-1} = \frac{4}{3} \left( \frac{1}{1 - \frac{4}{5}} \right) = \frac{20}{3}.
\]
(The series is geometric with \( a = \frac{4}{3} \) and \( r = \frac{4}{5} \).)

3. Discuss the convergence of the series
\[ \sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}} . \]

**Solution:** It converges conditionally. It’s an alternating series with \( b_n = 1/\sqrt{n} \).
We have (i) The sequence \( \{b_n\}_{n=2}^{\infty} \) is decreasing since \( \sqrt{n+1} > \sqrt{n} \) and thus \( b_{n+1} = 1/\sqrt{n+1} < 1/\sqrt{n} = b_n \) for all \( n \geq 2 \). (ii) \( \lim_{n \to \infty} b_n = \lim_{n \to \infty} 1/\sqrt{n} = 0 \). Thus the series converges by the Alternating Series Test. But the series
\[ \sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}} = \sum_{n=2}^{\infty} \frac{1}{\sqrt{n}} \]
diverges since it’s a \( p \) series and \( p = \frac{1}{2} < 1 \).

4. Use Comparison Tests to determine which one of the following series is divergent.

**Solution:**
(a) \( \sum_{n=1}^{\infty} \frac{1}{n^{2} + 1} \) converges by comparison with \( \sum_{n=1}^{\infty} \frac{1}{n^{2}} \), a \( p \)-series with \( p = \frac{2}{3} > 1 \).
(b) \( \sum_{n=1}^{\infty} \frac{1}{n^{2} + 8} \) converges by comparison with \( \sum_{n=1}^{\infty} \frac{1}{n^{2}} \), a \( p \)-series with \( p = 2 > 1 \).
(c) \( \sum_{n=1}^{\infty} \frac{n^{2} - 1}{n^{3} + 100} \) diverges by limit comparison with \( \sum_{n=1}^{\infty} \frac{1}{n} \), a \( p \)-series with \( p = 1 \).
(d) $\sum_{n=1}^{\infty} 7 \left( \frac{5}{6} \right)^n$ converges since it is a geometric series with $|r| = \frac{5}{6} < 1$.

(e) $\sum_{n=1}^{\infty} \frac{n}{n+1} \left( \frac{1}{2} \right)^n$ converges by comparison with $\sum_{n=1}^{\infty} \left( \frac{1}{2} \right)^n$, a geometric series with $|r| = \frac{1}{2} < 1$.

5. Which series below is the MacLaurin series (Taylor series centered at 0) for $\frac{x^2}{1+x}$?

Solution:

$$\frac{x^2}{1+x} = \frac{x^2}{1-(-x)} = x^2 \sum_{n=0}^{\infty} (-x)^n = \sum_{n=0}^{\infty} (-1)^n x^{n+2},$$

for $|x| < 1$.

6. Find the degree 3 MacLaurin polynomial (Taylor polynomial centered at 0) for the function $e^x \frac{1}{1-x^2}$

Solution: $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, and $\frac{1}{1-x^2} = \sum_{n=0}^{\infty} x^{2n}$. Thus writing only terms that will contribute, (that is terms of degree 3 or less)

$$\frac{e^x}{1-x^2} = e^x \cdot \frac{1}{1-x^2} = (1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots)(1 + x^2 + \cdots)$$

$$= 1 + x^2 + x + x^3 + \frac{x^2}{2} + \frac{x^3}{6} + \cdots = 1 + x + \frac{3}{2} x^2 + \frac{7}{6} x^3 + \cdots.$$

So the degree 3 Taylor polynomial is

$$T_3(x) = 1 + x + \frac{3}{2} x^2 + \frac{7}{6} x^3.$$

7. Which series below is a power series for $\cos(\sqrt{x})$?

Solution: Since $\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$, we have

$$\cos(\sqrt{x}) = \sum_{n=0}^{\infty} (-1)^n \frac{\sqrt{x}^{2n}}{(2n)!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{(2n)!} = 1 - \frac{x}{2!} + \frac{x^2}{4!} - \cdots.$$

8. Calculate

$$\lim_{x \to 0} \frac{\sin(x^3) - x^3}{x^9}.$$
Solution: Since \( \sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} \), we have

\[
\sin(x^3) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{6n+3}}{(2n+1)!} = x^3 - \frac{x^9}{3!} + \frac{x^{15}}{5!} - \cdots,
\]

and

\[
\lim_{x \to 0} \frac{\sin(x^3) - x^3}{x^9} = \lim_{x \to 0} \left( \frac{x^3 - \frac{x^9}{3!} + \frac{x^{15}}{5!} - \cdots}{x^9} \right) = -\frac{1}{6}.
\]

9. Does the series \( \sum_{n=1}^{\infty} \frac{(n!)^n}{n^{2n}} \) converge or diverge? Show your reasoning and state clearly any theorems or tests you are using.

Solution: Let \( a_n = \frac{(n!)^n}{n^{2n}} = \left( \frac{n!}{n^2} \right)^n \). Since

\[
\lim_{n \to \infty} \frac{n!}{n^2} = \lim_{n \to \infty} \frac{n-1}{n} \cdot (n-2)! = \infty,
\]

we have

\[
\lim_{n \to \infty} a_n = \lim_{n \to \infty} \left( \frac{n!}{n^2} \right)^n = \infty.
\]

Hence \( \lim_{n \to \infty} a_n \neq 0 \). By the Test for Divergence, the series is divergent.

Another possibility is to use the Root Test:

\[
\lim_{n \to \infty} \sqrt[n]{|a_n|} = \lim_{n \to \infty} \frac{n!}{n^2} = \infty.
\]

Since the limit is \( > 1 \), the series diverges.

10. Use the Integral Test to discuss whether the series \( \sum_{n=1}^{\infty} \frac{(\ln n)^2}{n} \) converges.

Solution: Let \( f(x) = \frac{(\ln x)^2}{x} \). It is continuous and positive when \( x > 1 \). Note

\[
f'(x) = \frac{2x \ln x \cdot \frac{1}{x} - (\ln x)^2}{x^2} = \frac{2 \ln x - (\ln x)^2}{x^2} = \frac{\ln x}{x^2}(2 - \ln x),
\]

and \( 2 - \ln x < 0 \), i.e., \( 2 < \ln x \), if \( x > e^2 \). Then \( f'(x) < 0 \) and hence \( f(x) \) is decreasing for \( x > e^2 \). Therefore, we can use the Integral Test. We compute the indefinite integral \( \int \frac{(\ln x)^2}{x} \, dx \) using the substitution \( u = \ln x \), getting

\[
\int \frac{(\ln x)^2}{x} \, dx = \int u^2 \, du = \frac{u^3}{3} + C = \frac{(\ln x)^3}{3} + C.
\]
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So
\[
\int_1^{\infty} \frac{(\ln x)^2}{x} \, dx = \lim_{t \to \infty} \int_1^t \frac{(\ln x)^2}{x} \, dx = \lim_{t \to \infty} \left( \frac{(\ln x)^3}{3} \right)_1^t = \lim_{t \to \infty} \frac{(\ln t)^3}{3} = \infty.
\]

Hence the improper integral \( \int_1^{\infty} \frac{(\ln x)^2}{x} \, dx \) is divergent. By the Integral Test, the series \( \sum_{n=1}^{\infty} \frac{(\ln n)^2}{n} \) diverges. (Note: It is easier to prove this series diverges using the Comparison Test, comparing to the harmonic series. But we were not at liberty to use this test.)

11. Find the radius of convergence and interval of convergence of the power series
\[
\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}} (x - 3)^n
\]

**Solution:** Set \( a_n = \frac{(-1)^n}{\sqrt{n}} (x - 3)^n \). Using the Ratio Test,
\[
\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \frac{\sqrt{n}}{\sqrt{n + 1}} \left| x - 3 \right| = \left| x - 3 \right|.
\]

Hence, the radius of convergence is 1, and the series converges absolutely for \( |x - 3| < 1 \), or \( 2 < x < 4 \). For the end points, when \( x = 2 \), the series is \( \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \) which is divergent since it is a \( p \)-series with \( p = \frac{1}{2} < 1 \); when \( x = 4 \), the series is \( \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}} \) which is convergent since it’s an alternating series, and \( b_n = \frac{1}{\sqrt{n}} \) is decreasing and \( \lim_{n \to \infty} \frac{1}{\sqrt{n}} = 0 \). (See the solution to Problem #3 for details.) Hence, the interval of convergence is \( 2 < x \leq 4 \).

12. (a) Show that
\[
\sum_{n=0}^{\infty} (-1)^n x^{2n} = \frac{1}{1 + x^2}
\]
provided that \( |x| < 1 \).

(b) Find
\[
\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n + 1)(\sqrt{3})^{2n+1}}.
\]

**Solution:** (a) Since \( |x| < 1 \), we have \( |x^2| < 1 \). Hence
\[
\frac{1}{1 + x^2} = \frac{1}{1 - (-x^2)} = \sum_{n=0}^{\infty} (-x^2)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n}.
\]
(b) Integrate both the left and right hands of (a) to get
\[
\int \sum_{n=0}^{\infty} (-1)^n x^{2n} \, dx = \int \frac{1}{1 + x^2} \, dx
\]
\[\Rightarrow \sum_{n=0}^{\infty} (-1)^n x^{2n} \, dx = \int \frac{1}{1 + x^2} \, dx
\]
\[\Rightarrow \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n + 1} = \arctan x + C.
\]
Letting \(x = 0\), we have \(C = 0\). Hence, we have
\[
\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n + 1} = \arctan x.
\]
Let \(x = \frac{1}{\sqrt{3}}\). We get
\[
\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n + 1)(\sqrt{3})^{2n+1}} = \arctan \frac{1}{\sqrt{3}} = \frac{\pi}{6}.
\]