Math 10850, fall 2019

Final exam, Monday December 16

NAME:

Instructions

- The exam goes from 4.15pm to 6.15pm.
- There are 8 questions, plus a take-home question (Question 9), and a bonus question.
- The bonus question is mostly for entertainment (but also for a few points). Please only attempt it if you are sure that you have nothing left to say on the rest of the exam!
- Present your answers in the space provided. Use the back of each page if necessary; if you do, clearly indicate this.
- Present your answers clearly and neatly.
- **Justify all your assertions**, even if a question does not explicitly say this. Partial credit can be given, but only if your answers are supported. Plus, the *justification* of mathematical assertions has been the main point of this course.
- Calculators are not allowed, nor should they be needed. No notes, books or any other external resources are allowed, either.
- Remember the Academic Code of Honor Pledge:

> "As a member of the Notre Dame community, I acknowledge that it is my responsibility to learn and abide by principles of intellectual honesty and academic integrity, and therefore I will not participate in or tolerate academic dishonesty."

MAY THE FORCE BE WITH YOU!

Question	score	out of
1		8
2		8
3		11
4		8
5		8
6		9
7		8
8		5
Bonus		2
9		2
Total		67

1. (8 points)

   (a) (3 points) Give a precise definition (in symbols or words) of

      "The function f approaches a limit near a".

   (b) (5 points) Show (using the definition of limit) that if $\lim_{x\to a} f(x) = L$, and c is any real number, then $\lim_{x\to a}(cf)(x) = cL$.

2. (8 points)

   (a) (3 points) Give the definition of a function f being *continuous* at a. (This doesn't have to be an ε-δ definition, but it has to be precise.)

   (b) (5 points) Suppose that f is differentiable at a. Show that it is also continuous at a.

3. (11 points)

(a) (3 points) *Directly from the definition of the derivative*, find $f'(x)$ for all $x \neq -2$, where $f(x) = \frac{2}{2+x}$. (You can assume reasonable facts about limits).

(b) (3 points) Using any properties of derivatives/differentiation that you know, compute $f''(x)$, $f'''(x)$ and $f^{(4)}(x)$ (i.e., $f''''(x)$).

(c) (5 points) Based on your answer to part (b), conjecture a general expression for $f^{(n)}(x)$, and use induction to prove that it is correct. (It might be helpful to remember the factorial function, $n! = n \cdot (n-1) \cdot (n-2) \cdot \dots \cdot 3 \cdot 2 \cdot 1$.)

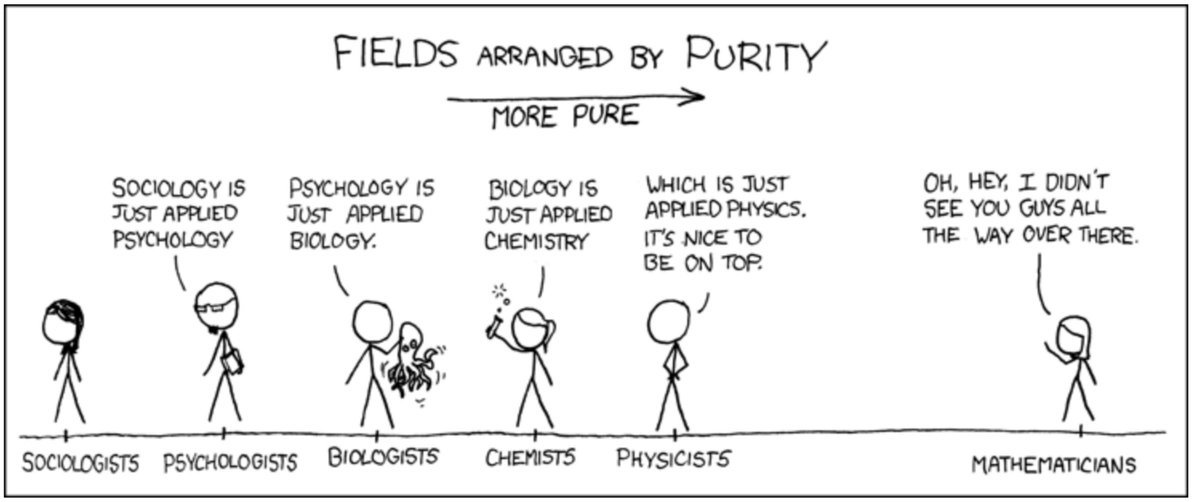
4. (8 points)

(a) (4 points) Suppose that $f(x) \geq 0$ for all x and that $\lim_{x\to a} f(x)$ exists. Prove that $\lim_{x\to a} f(x) \geq 0$.

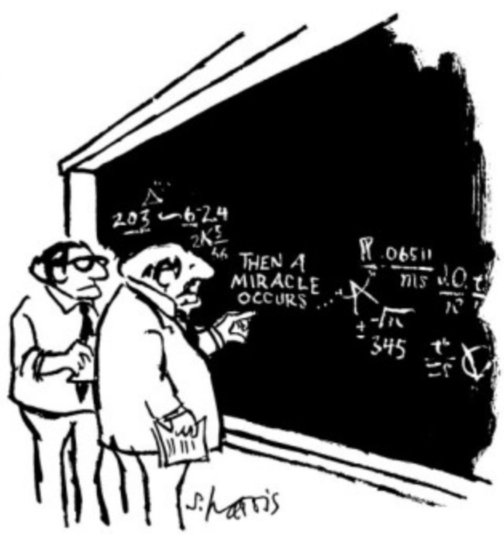
(b) (4 points) Suppose that $f(x) > 0$ for all x and that $\lim_{x\to a} f(x)$ exists. Either justify the following statement, or give an example to show that is is false:

$$\lim_{x\to a} f(x) > 0.$$

(You can be more relaxed about ε-δ details in this part. E.g., it would ok to say "Just as in part (a)" without giving rigorous details, or to assert obvious limits without giving rigorous details.)



XKCD by Randall Munroe



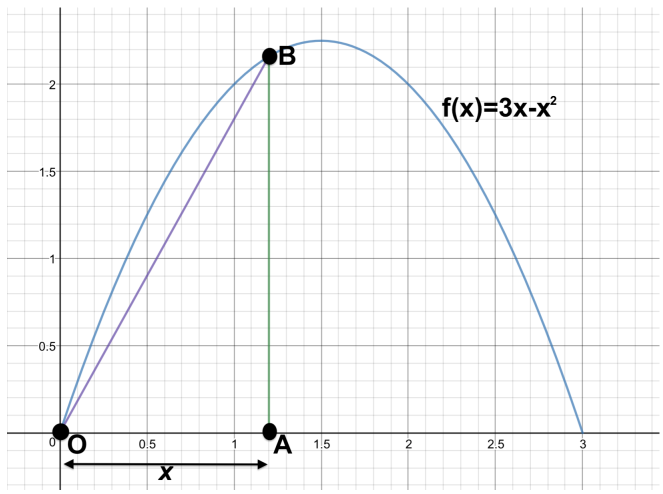
"I think you should be more explicit here in step two."

By S. Harris

5. (8 points)

(a) (3 points) Let f be a function defined on (a, b). What does the Fermat principle say about local maxima and local minima of f on (a, b)?

(b) (5 points) The figure below shows the graph of the function $f : [0, 3] \to \mathbb{R}$ given by $f(x) = 3x - x^2$. The point A is on the x-axis, distance x from the origin O (with x in the domain of f), and the angle OAB is a right angle. Find the choice of x that maximizes the area $A(x)$ of the triangle OAB, *carefully justifying your choice.* (The careful justification is the main point of this question.)



6. (9 points) Give clear and correct statements of each of the three main theorems of the semester:

   (a) Intermediate Value Theorem:

   (b) Extreme Value Theorem:

   (c) Mean Value Theorem:

7. (8 points)

(a) (3 points) State the definition of $\lim_{x\to\infty} f(x) = \infty$.

(b) (5 points) Show that if both $\lim_{x\to\infty} f(x) = \infty$ and $\lim_{x\to\infty} g(x) = \infty$, then $\lim_{x\to\infty}(f+g)(x) = \infty$.

8. (5 points) Prove that for all real numbers x, y,

$$|\sin x - \sin y| \leq |x - y|.$$

Bonus question; *please* don't think about this unless you have free time at the end of the exam! (2 points)

Recall that in an early homework you showed that $1+x+x^2 > 0$ always, and that $1+x+x^2+x^3+x^4 > 0$ always. This problem asks you to generalize this.

Let n be an *even* natural number. Prove that

$$1 + x + x^2 + x^3 + \cdots + x^{n-1} + x^n$$

is strictly positive for all real x. Furthermore, show f has just one local minimum (which is also a global minimum) and no local maxima.

9. (2 points) Have a peaceful, restful and happy Christmas. (Since grades are due before Christmas, I'll give you full credit for this question in advance of the fact).

Peanuts by Charles Schultz