Math 10850, Honors Calculus 1

Homework 3

Solutions

1. Express each of the following without absolute value signs, treating various cases separately where necessary. Try to use as few cases as possible. Write your final solution using the *brace notation*, for example

$$\text{this thing} = \begin{cases} \text{something} & \text{if condition/case 1} \\ \text{something else} & \text{if condition 2} \\ \text{something else again} & \text{if condition 3} \end{cases}$$

   (a) $|a+b| - |b|$ (where a, b might be any real numbers).

   Solution: If $a, b \le 0$ then $|a+b| - |b| = -(a+b) + b = -a$, and if $a, b \ge 0$ then $|a+b| - |b| = (a+b) - b = a$.

   If $a \le 0 \le b$, then the status of $|a+b|$ depends on whether $-a \le b$, in which case $|a+b| - |b| = (a+b) - b = a$, or $-a \ge b$, in which case $|a+b| - |b| = -(a+b) - b = -a - 2b$.

   If $b \le 0 \le a$, then the status of $|a+b|$ depends on whether $-b \le a$, in which case $|a+b| - |b| = (a+b) + b = a + 2b$, or $-b \ge a$, in which case $|a+b| - |b| = -(a+b) + b = -a$.

   This covers all cases, so we have:

$$|a+b| - |b| = \begin{cases} -a & \text{if } a, b \le 0 \text{ or if } b \le 0 \le a \text{ and } -b \ge a \\ a & \text{if } a, b \ge 0 \text{ or if } a \le 0 \le b \text{ and } -a \le b \\ -a - 2b & \text{if } a \le 0 \le b \text{ and } -a \ge b \\ a + 2b & \text{if } b \le 0 \le a \text{ and } -b \le a. \end{cases}$$

   (b) $a - |(a - |a|)|$ (where a might be any real number).

   Solution: If $a \ge 0$ then $|a| = a$, so $a - |a| = 0$. If $a \le 0$ then $|a| = -a$, so $a - |a| = 2a$, and since this is negative, $|a - |a|| = -2a$

$$a - |a - |a|| = \begin{cases} a & \text{if } a \ge 0 \\ 3a & \text{if } a \le 0. \end{cases}$$

2. Find all real numbers x for which:

(a) $|x+4|<2$.

Solution: Case 1, $x+4 \geq 0$, so $x \geq -4$. In this case $|x+4|<2$ is equivalent to $x+4<2$ or $x<-2$. The numbers x that are ≥ -4 and <-2 are the numbers between -4 and -2, including -4 but not including -2.

Case 2, $x+4 \leq 0$, so $x \leq -4$. In this case $|x+4|<2$ is equivalent to $-x-4<2$ or $x>-6$. The numbers x that are ≤ -4 and >-6 are the numbers between -6 and -4, including -4 but not including -6.

Combining the cases, we find that the set of x satisfying $|x+4|<2$ is also the set of x satisfying

$$-6<x<-2.$$

Alternatively, and more easily: what does $|y|<a$ mean? It means that the distance from y to 0 is smaller than a, which happend exactly if $-a<y<a$ (as could easily be verified by a case analysis). So $|x+4|<2$ encodes all those x's such that the distance from $x+4$ to 0 is smaller than 2, i.e. the distance from x to -4 is smaller than 2, i.e. the numbers x that are (strictly) within distance 2 of -4, i.e. the numbers strictly between -6 and -2.

Succinctly: $|x+4|<2$ means $-2<x+4<2$ or $-6<x<-2$.

(b) $|x-1|+|x+1|<2$.

Solution: We treat cases.

i. If $x \geq 1$ then $|x-1|+|x+1|=x-1+x+1=2x$, and $2x<2$ means $x<1$. This is inconsistent with $x \geq 1$, so there is no valid x in this range.

ii. If $-1 \leq x \leq 1$ then $|x-1|+|x+1|=1-x+x+1=2$, so in this range we cannot have $|x-1|+|x+1|<2$.

iii. If $x \leq -1$ then $|x-1|+|x+1|=1-x-x-1=-2x$, and $-2x<2$ means $x>-1$. This is inconsistent with $x \leq -1$, so there is no valid x in this range.

Conclusion: No x satisfies the given inequality.

(c) $|x-1| \cdot |x+2|=3$.

Solution: $|x-1||x+2|=3$. Again we treat cases.

i. If $x \geq 1$ then $|x-1||x+2|=(x-1)(x+2)$, and the equation becomes $x^2+x-5=0$. This quadratic has two real solutions, one of which, $(\sqrt{21}-1)/2$, lies in the range currently under consideration, so this is a valid x.

ii. If $-2 \leq x \leq 1$ then $|x-1||x+2|=(1-x)(x+2)$, and the equation becomes $-x^2-x-1=0$. This quadratic has no real solutions.

iii. If $x \leq -2$ then $|x-1||x+2|=(1-x)(-x-2)$, and the equation becomes $x^2+x-5=0$. This quadratic has two real solutions, one of which, $(-\sqrt{21}-1)/2$, lies in the range currently under consideration, so this is a valid x.

Conclusion: the two x that satisfy the equation are $x=(\pm\sqrt{21}-1)/2$.

3. Prove each of the following inequalities (if you think it might be useful, you can assume the triangle inequality):

(a) $|x - y| \leq |x| + |y|$ for all reals x, y.

Solution: One possible approach is through cases:

- Case 1: $x \geq 0$, $y \geq 0$, $x \geq y$.
- Case 2: $x \geq 0$, $y \geq 0$, $x \leq y$.
- Case 3: $x \leq 0$, $y \leq 0$, $x \geq y$.
- Case 4: $x \leq 0$, $y \leq 0$, $x \leq y$.
- Case 5: $x \geq 0$, $y \leq 0$ (no need for a further splitting into $x \geq y$ or $x \leq y$ here, since in this case $x - y \geq 0$ is forced), and
- Case 6: $x \leq 0$, $y \geq 0$ (no need for a further splitting into $x \geq y$ or $x \leq y$ here).

Inside each case, the absolute value signs can be consistently unpacked. E.g., for Case 3, the inequality becomes

$$x - y \leq -x - y,$$

which is obviously true; and in Case 6 the inequality becomes

$$-x + y \leq -x + y,$$

which is also obviously true.

Here's a much quicker proof: By the triangle inequality applied to x and $-y$, we get

$$|x + (-y)| \leq |x| + |-y|$$

or

$$|x - y| \leq |x| + |-y|.$$

But now $|-y| = |y|$ (an easy case analysis: if $y \geq 0$ then both sides are y, and if $y \leq 0$ both sides are negative y). So we conclude that indeed

$$|x - y| \leq |x| + |y|.$$

(b) $|x| - |y| \leq |x - y|$ for all reals x, y.

Solution: Again we could do a case analysis; but much simpler is to apply the triangle inequality with $a = x - y$ and $b = y$. From

$$|a + b| \leq |a| + |b|$$

we deduce

$$|(x - y) + y| \leq |x - y| + |y|$$

or

$$|x| - |y| \leq |x - y|.$$

4. The maximum of two numbers x, y is denoted $\max\{x, y\}$, and the minimum is denoted $\min\{x, y\}$. So, for example,

- $\max\{3, -1\} = 3$
- $\min\{4.5, 4.5\} = 4.5$
- $\min\{-3, -4\} = -4$.

(a) Prove that $\max\{x, y\} = \frac{x+y+|y-x|}{2}$.

Solution: We treat cases. If $x \geq y$ then

- $\max\{x, y\} = x$ and
- $\frac{x+y+|y-x|}{2} = \frac{x+y+-(y-x)}{2} = \frac{2x}{2} = x$,

so in this case $\max\{x, y\} = \frac{x+y+|y-x|}{2}$.
On the other hand, if $y \geq x$ then

- $\max\{x, y\} = y$ and
- $\frac{x+y+|y-x|}{2} = \frac{x+y+(y-x)}{2} = \frac{2y}{2} = y$,

so in this case again $\max\{x, y\} = \frac{x+y+|y-x|}{2}$.
In all cases, $\max\{x, y\} = \frac{x+y+|y-x|}{2}$, as claimed.

(b) Find a very similar formula for $\min\{x, y\}$.

Solution: I claim that

$$\min\{x, y\} = \frac{x + y - |y - x|}{2}.$$

The proof, a similar case analysis to the previous part, is omitted.

(c) Find a formula for $\max\{x, y, z\}$ (the maximum of the three numbers x, y and z). You can use x, y, z, addition, subtraction, division, multiplication, and absolute value, but **not** max or min.

Solution: On way is to use that $\max\{a, b, c\} = \max\{a, \max\{b, c\}\}$, and piggy-back off the formula from the first part of the question:

$$\begin{aligned}
\max\{x, y, z\} &= \max\{x, \max\{y, z\}\} \\
&= \max\{x, \frac{y + z + |z - y|}{2}\} \\
&= \frac{x + \frac{y+z+|z-y|}{2} + \left|\frac{y+z+|z-y|}{2} - x\right|}{2}.
\end{aligned}$$

It's ugly, but it works!

(d) Define middle$\{x, y, z\}$ to be the middle number of x, y and z when the three are written in increasing order (so, for example, middle$\{7, -8, 2\}$ is 2, and middle$\{0, 0, 1\}$ is 0). Find a formula for middle$\{x, y, z\}$ that only uses x, y, z, addition, subtraction, division, multiplication, and absolute value.

Solution: Arguing similarly to the previous part of the question, we get

$$\min\{x,y,z\} = \frac{x + \frac{y+z-|z-y|}{2} - \left|\frac{y+z-|z-y|}{2} - x\right|}{2}.$$

But now we can use

$$\text{middle}\{x,y,z\} = x+y+z-\max\{x,y,z\}-\min\{x,y,z\}$$

to get $\text{middle}\{x,y,z\} =$

$$x+y+z-\left(\frac{x+\frac{y+z+|z-y|}{2}+\left|\frac{y+z+|z-y|}{2}-x\right|}{2}\right)-\left(\frac{x+\frac{y+z-|z-y|}{2}-\left|\frac{y+z-|z-y|}{2}-x\right|}{2}\right).$$

It's even uglier, but it also works!

5. Although it is not immediately apparent, this question is related to the fact that if $a \neq 0$ then $a^2 > 0$ (that's a hint).

   (a) Find the smallest possible value of $2x^2 - 3x + 4$ as x runs over real numbers.

   Solution: Write

   $$2x^2 - 3x + 4 = 2(x^2 - (3/2)x + 2) = 2\left(\left(x - \frac{3}{4}\right)^2 + \frac{23}{16}\right).$$

   Since

   $$\left(x - \frac{3}{4}\right)^2 \geq 0$$

   the whole expression is always at least $23/8$; and in fact it can equal $23/8$, at $x = 3/4$. So the smallest possible value is $23/8 = 2.875$.

   (b) Find the smallest possible value of $x^2 - 3x + 2y^2 + 4y + 2$ as x and y run over real numbers.

   Solution: Play the same game as before! Write

   $$x^2 - 3x + 2y^2 + 4y + 2 = \left(x - \frac{3}{2}\right)^2 + 2(y+1)^2 - \frac{9}{4}.$$

   The minimum is $-9/4$, reached at $x = 3/2$ and $y = -1$.

   (c) Find the smallest possible value of $x^2 + 4xy + 5y^2 - 4x - 6y + 7$ as x and y run over real numbers.

   Solution: This is rather harder, because of the interaction of x and y in the $4xy$ term. But still, we can write

   $$\begin{aligned} x^2 + 4xy + 5y^2 - 4x - 6y + 7 &= x^2 + 4(y-1)x + 5y^2 - 6y + 7 \\ &= (x + 2(y-1))^2 - 4(y-1)^2 + 5y^2 - 6y + 7 \\ &= (x + 2(y-1))^2 + (y+1)^2 + 2. \end{aligned}$$

Now it is clear that the minimum is 2, achieved at $y = -1$ and

$$x + 2(-1 - 1) = 0$$

or $x = 4$.

6. (Here, if your answer to a part is "yes" then you need to give a proof; if your answer is "no" you need to give an example that illustrates this. A *rational* number is a number that can be expressed as the ratio of two whole numbers. A number which is not rational is *irrational*. Remember that in class we proved that $\sqrt{2}$ is a real number that is irrational.)

   (a) If a is rational and b irrational, is $a + b$ necessarily irrational?

   Solution: If a is rational and b irrational, then $a + b$ must be irrational; for if $a + b$ were rational, then so would $(a + b) - a = b$, a contradiction.

   (b) If a is irrational and b irrational, is $a + b$ necessarily irrational?

   Solution: If a and b are both irrational, then $a + b$ is not necessarily irrational; for example, $-\sqrt{2}$ and $\sqrt{2}$ are both irrational, but their sum is 0. [It *might* be irrational: consider $a = b = \sqrt{2}$]

   (c) If a is rational and b irrational, is ab is necessarily irrational?

   Solution: If a is rational and b irrational, then ab is not necessarily irrational; for example, if $a = 0$ then $ab = 0$ which is rational. [If a is *not* 0, then ab must be irrational; for if ab were rational, then so would $a^{-1}(ab) = b$, a contradiction]

   (d) Is there a number a such that a^2 is irrational but a^4 is rational?

   Solution: There is such a number. For example, $\sqrt{\sqrt{2}}$ is irrational (if it were rational then its square would be too, but its square is $\sqrt{2}$), and $(\sqrt{\sqrt{2}})^4 = 2$.

   (e) Does there exist two irrational numbers whose sum and product are both rational?

   Solution: Yes! For example, $-\sqrt{2}$ and $\sqrt{2}$.

7. (a) In class we used a "parity trick" (looking at oddness and evenness of a proposed numerator and denominator) to prove that $\sqrt{2}$ is not rational. Use a variant of this trick to prove that $\sqrt{6}$ is irrational. (One possibility is to think about remainder on division by 6).

   Solution: For all natural numbers n there is an integer m such that (exactly) one of $n = 6m$ or $n = 6m + 1$ or $n = 6m + 2$ or $n = 6m + 3$ or $n = 6m + 4$ or $n = 6m + 5$ holds (an "obvious" fact, that can be properly proven by a tedious but easy induction). It is tedious but easy to check that n is of the form $6m$ if and only if n^2 is of that form. (If $n = 6m + 1$ then $n^2 = 6m' + 1$; if $n = 6m + 2$ then $n^2 = 6m' + 4$; if $n = 6m + 3$ then $n^2 = 6m' + 3$; if $n = 6m + 4$ then $n^2 = 6m' + 2$; if $n = 6m + 5$ then $n^2 = 6m' + 1$).

Suppose there are natural numbers a, b with $(a/b)^2 = 6$. We may assume (by dividing a, b by 6 repeatedly) that at least one of a, b is not divisible by 6. Now $a^2 = 6b^2$, so a^2 is of the form $6m$, so a is of the form $6m'$ for some integer m', which must in fact be a natural number — it certainly isn't negative, nor is it 0, else a and so a/b is 0, so $(a/b)^2 \neq 6$.

It follows that $36(m')^2 = 6b^2$, so $6(m')^2 = b^2$, so b^2 is a multiple of 6, so b is also, a contradiction of the assumption that at least one of a, b is not a multiple of 6.

We conclude that no natural numbers a, b exist with $(a/b)^2 = 6$, and so $\sqrt{6}$ is irrational.

(b) Prove that $\sqrt[3]{2}$ is irrational.

Solution: All natural numbers n are either even or odd. It is easy to check that n is even if and only if n^2 is even.

Suppose there are natural numbers a, b with $(a/b)^3 = 2$. We may assume (by dividing a, b by 2 repeatedly) that at least one of a, b is odd. Now $a^3 = 2b^3$, so a^3 is even, so a is even, say $a = 2m$ for some integer m, which must in fact be a natural number — it certainly isn't negative, nor is it 0, else a and so a/b is 0, so $(a/b)^3 \neq 2$.

It follows that $8m^4 = 2b^3$, so $4m^3 = b^2$, so b^2 is even, so b is also, a contradiction of the assumption that at least one of a, b is odd.

We conclude that no natural numbers a, b exist with $(a/b)^3 = 2$, and so $\sqrt[3]{2}$ is irrational.

(c) Prove that $\sqrt{2} + \sqrt{3}$ is irrational.

Solution: Suppose that $\sqrt{2} + \sqrt{3}$ is rational. Then so is $(\sqrt{2} + \sqrt{3})^2 = 2 + 2\sqrt{2}\sqrt{3} + 3 = 5 + 2\sqrt{6}$, which implies that $\sqrt{6}$ is rational, a contradiction (using the first part of the question).