Math 10850, Honors Calculus 1

Homework 4

Solutions

1. Prove the following identities. The main point here is that you should be working towards laying out your proof in a clear and organized manner. Use the proof from class that $1 + 2 + \ldots + n = n(n+1)/2$ as a template.

   - Begin the proof by saying that it will be a proof by induction on n.
   - Verify the base case, and when you do so, clearly indicate that that is what you are doing
   - When you move onto to the induction step, clearly indicate that that is what you are doing.
   - In the induction step, explicitly state what you are assuming (the inductive hypothesis), and then clearly deduce what you want to deduce.
   - End with a concluding statement, along the lines of "By induction, we conclude that ...".)

   (a) For all natural numbers n,

   $$\sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2.$$

   Note that this says: the sum of the *cubes* of the first n numbers, is the same as the *square* of the sum of the first n numbers; an odd fact!

   Solution: We prove $1^3 + 2^3 + \ldots + n^3 = (1 + 2 + \ldots + n)^2$ by induction on n.

   Base case $n = 1$: We assert $1^3 = 1^2$, which is true.

   Induction step: We assume that for some $n \geq 1$, $1^3 + 2^3 + \ldots + n^3 = (1 + 2 + \ldots + n)^2$. We have

   $$1^3 + 2^3 + \ldots + n^3 + (n+1)^3 = (1 + 2 + \ldots + n)^2 + (n+1)^3 \quad \text{(induction hypotheses)}.$$

   We want to argue that the right-hand side above equals $(1 + 2 + \ldots + n + (n+1))^2$. We have $(1 + 2 + \ldots + n + (n+1))^2$

   $$\begin{aligned}
   &= (1 + 2 + \ldots + n)^2 + 2(n+1)(1 + 2 + \ldots + n) + (n+1)^2 \quad \text{using } (x+y)^2 = x^2 + 2xy + y^2 \\
   &= (1 + 2 + \ldots + n)^2 + 2(n+1)\frac{n(n+1)}{2} + (n+1)^2 \quad \text{using } 1 + 2 + \ldots + n = n(n+1)/2, \text{wh} \\
   &= (1 + 2 + \ldots + n)^2 + n(n+1)^2 + (n+1)^2 \\
   &= (1 + 2 + \ldots + n)^2 + (n+1)^3.
   \end{aligned}$$

So $1^3 + 2^3 + \ldots + n^3 + (n+1)^3 = (1 + 2 + \ldots + n + (n+1))^2$, and we conclude by induction that $1^3 + 2^3 + \ldots + n^3 = (1 + 2 + \ldots + n)^2$ for all $n \geq 1$.

(b) Remember that the Fibonacci numbers are defined by the recurrence relation

$$f_n = \begin{cases} 0 & \text{if } n = 0 \\ 1 & \text{if } n = 1 \\ f_{n-1} + f_{n-2} & \text{if } n \geq 2. \end{cases}$$

Prove that for all $n \geq 0$,

$$\sum_{k=0}^{n} f_k^2 = f_n f_{n+1}.$$

Solution: We proceed by induction on n, with the base case $n = 0$ being obvious. For $n \geq 0$, assuming $\sum_{k=0}^{n} f_k^2 = f_n f_{n+1}$ we get

$$\begin{aligned} \sum_{k=0}^{n+1} f_k^2 &= f_{n+1}^2 + \sum_{k=0}^{n+1} f_k^2 \\ &= f_{n+1}^2 + f_n f_{n+1} \\ &= f_{n+1}(f_{n+1} + f_n) \\ &= f_{n+1} f_{n+2} \\ &= f_{(n+1)+1} f_{(n+1)}, \end{aligned}$$

and we are done by induction.

(c) For all natural numbers n,

$$\sum_{k=1}^{n} (3k^2 - 3k + 1) = ???.$$

(Here I'll leave it up to you to find the correct right-hand side — a simple expression that doesn't involve a sum — and then prove that what you have found is correct)

Solution: By checking a few small values of n, one can spot the pattern that $\sum_{i=1}^{n} (3k^2 - 3k + 1) = n^3$, and then prove this, in a straightforward way, by induction.

Base case ($n = 1$): this assets that $(3(1)^2 - 3(1) + 1) = 1^3$, which is true.

Induction step: Suppose, for some $n \geq 1$, that

$$\sum_{k=1}^{n} (3k^2 - 3k + 1) = n^3.$$

We have

$$\begin{aligned} \sum_{k=1}^{n+1} (3k^2 - 3k + 1) &= \left(\sum_{k=1}^{n} (3k^2 - 3k + 1) \right) + (3(n+1)^2 - 3(n+1) + 1) \\ &= n^3 + (3n^2 + 3n + 1) \quad \text{(induction hypothesis and algebra)} \\ &= (n+1)^3 \quad \text{(binomial theorem, or algebra.)} \end{aligned}$$

So the identity is proven by induction.

2. (a) Let r be a real number that's not equal to 1. Prove by induction on n that

$$1 + r + r^2 + \ldots + r^n = \frac{1 - r^{n+1}}{1 - r}.$$

Solution (sketch; see earlier for examples of fully written out induction proofs): The base case $n = 1$ is the assertion that for all $r \neq 1$, $1 + r = (1 - r^2)/(1 - r)$, which is easy to verify.

For the induction step:

$$\begin{aligned} 1 + r + r^2 + \ldots + r^n + r^{n+1} &= \frac{1 - r^{n+1}}{1 - r} + r^{n+1} \\ &= \frac{1 - r^{n+1} + (1 - r)r^{n+1}}{1 - r} \\ &= \frac{1 - r^{n+1} + r^{n+1} - r^{n+2}}{1 - r} \\ &= \frac{1 - r^{n+2}}{1 - r}. \end{aligned}$$

(b) Set

$$S = 1 + r + r^2 + \ldots + r^n.$$

By multiplying both sides by r and doing some algebraic manipulation on the two equations, give a different (non-induction) proof of the result from the part (a).

Solution, part (b): Let

$$S = 1 + r + r^2 + \ldots + r^n.$$

We have

$$rS = r + r^2 + \ldots + r^n + r^{n+1},$$

and so

$$(1 - r)S = S - rS = 1 + r + r^2 + \ldots + r^n - (r + r^2 + \ldots + r^n + r^{n+1}) = 1 - r^{n+1}$$

(the last equality by rearranging terms to cancel r with $-r$, r^2 with $-r^2$, etc.). Dividing through by $1 - r$ (which is valid since $r \neq 1$) we get

$$S = \frac{1 - r^{n+1}}{1 - r}.$$

3. In class we defined the expression a^n for all real a and all natural numbers n, via a recursive definition. Prove (by induction) that for all natural numbers n and m we have

$$a^{n+m} = a^n a^m.$$

(Hint: don't try to be too fancy with the induction; pick either induction on n or induction on m, but not both at once.)

Solution: Let $p(m)$ be the predicate "For all real a, and for all Natural numbers $n \geq 1$, $a^{n+m} = a^n a^m$". We prove $p(m)$ for all Natural numbers $m \geq 1$ by induction on m.

Base case $n = 1$: We want to show $a^{n+1} = a^n a$ for all $n \geq 1$ and for all real a. Using commutativity ($a^n a = aa^n$) this becomes simply the definition of a^{n+1}, so is true.

Induction step: We assume that for some $m \geq 1$, $p(m)$ is true. We want to argue that $p(m+1)$ is true. Let a be a real number, and n a Natural number. We want to argue that $a^{n+(m+1)} = a^n a^{m+1}$. By associativity (first equality), definition (second equality), induction hypothesis (third equality), commutativity and associativity (fourth equality) and definition (fifth equality)

$$a^{n+(m+1)} = a^{(n+m)+1} = aa^{n+m} = a(a^n a^m) = a^n(aa^m) = a^n a^{m+1},$$

as required. Since this argument was for all real a and all Natural numbers n, we have verified that $p(m)$ implies $p(m+1)$. By induction, $p(m)$ is true for all $m \geq 1$.

4. Prove that if p, q are rational numbers, $x = p + \sqrt{q}$, and m is a natural number, then $x^m = a + b\sqrt{q}$ for some rational numbers a, b.

Solution: We are given that p and q are rational numbers. Let $p(m)$ be the predicate "$(\exists a)(\exists b)((a \in \mathbb{Q}) \wedge (b \in \mathbb{Q}) \wedge ((p+\sqrt{q})^m = a+b\sqrt{q}))$". We prove $(\forall m)p(m)$ by induction on m.

The base case $m = 1$ is easy: we may take $a = p$ and $b = 1$.

For the induction step, suppose that $(p + \sqrt{q})^m = a + b\sqrt{q}$ for some rationals a and b. Then

$$\begin{aligned} (p+\sqrt{q})^{m+1} &= (p+\sqrt{q})(p+\sqrt{q})^m \\ &= (p+\sqrt{q})(a+b\sqrt{q}) \\ &= ap + bp\sqrt{q} + a\sqrt{q} + b\sqrt{q}\sqrt{q} \\ &= (ap+bq) + (a+bp)\sqrt{q}. \end{aligned}$$

Since $ap + bq, a + bp$ are both rational, we infer $p(m+1)$ from $p(m)$, and so we conclude $(\forall m)p(m)$ by induction.

Here is an alternate solution: By the binomial theorem,

$$(p+\sqrt{q})^m = \binom{m}{0} p^m (\sqrt{q})^0 + \binom{m}{1} p^{m-1} (\sqrt{q})^1 + \cdots + \binom{m}{m-1} p^1 (\sqrt{q})^{m-1} + \binom{m}{m} p^0 (\sqrt{q})^m.$$

There are two kinds of term in this sum:

- $\binom{m}{k} p^{m-k} (\sqrt{q})^k$ where k is even. Each one of these terms is rational: it is $\binom{m}{k} p^{m-k} q^{k/2}$. Gathering (adding) all these terms together gives a rational number a.

- $\binom{m}{k}p^{m-k}(\sqrt{q})^k$ where k is odd. Each one of these terms is a rational number multiplied by $\sqrt{q}$: it is $\binom{m}{k}p^{m-k}q^{(k-1)/2}\sqrt{q}$. Gathering (adding) all these terms together gives a $b\sqrt{q}$ for some rational number b.

5. Identify[1] the error in the following proof of the claim "All cows are the same color":

   Let $p(n)$ be the predicate "any n cows are the same color". We prove that $p(n)$ is true for all $n \geq 1$ (and so that all cows are the same color), by induction on n.

   Base case $n = 1$: any one cow is a set of cows all of which are the same color (whatever color the cow under consideration is). So $p(1)$ is true.

   Induction step: Suppose that for some $n \geq 1$, $p(n)$ is true. Let

   $$\{\mathrm{Cow}_1, \mathrm{Cow}_2, \ldots, \mathrm{Cow}_n, \mathrm{Cow}_{n+1}\}$$

   be a set of $n+1$ cows. By the induction hypothesis (the fact that $p(n)$ is True), all of Cow_1, Cow_2, …, Cow_n are the same color; call that color C. Also by the induction hypothesis, all of Cow_2, Cow_3, …, Cow_n, Cow_{n+1} are the same color (this is another collection of n cows). That common color must be C, because Cow_2 (for example) is colored C, from the first application of induction hypothesis. It follows that all of Cow_1, Cow_2, …, Cow_n, Cow_{n+1} are the same color, C, and so $p(n+1)$ is True.

   By induction, we conclude that $p(n)$ is True for all $n \geq 1$, and so all cows are the same color.

   Solution: The error is the following: it is not necessarily the case that there is a cow in common to the sets

   $$\mathrm{Cow}_1,\ \mathrm{Cow}_2,\ \ldots,\ \mathrm{Cow}_n$$

   and

   $$\mathrm{Cow}_2,\ \mathrm{Cow}_3,\ \ldots,\ \mathrm{Cow}_n,\ \mathrm{Cow}_{n+1}.$$

   Specifically, if $n = 1$ then the first set consists of only Cow_1, while the second set consists of only Cow_2. Each set consists of a set of cows that are the same color, but because there is no cow in common to the sets, we cannot conclude that both cows are the same color.

   If $n > 1$ then the argument is fine. So this is a situation where $p(1)$ is True, and the truth of $p(n)$ implies the truth of $p(n+1)$ for all n **except** $n = 1$; this is enough for induction to fail.

6. The Fibonacci numbers (defined in question 1) are very closely related to the *golden ratio*, the number $(1+\sqrt{5})/2 \approx 1.618$, that is often denoted φ.

   (a) Prove (most easily by induction on n) that for $n \geq 1$,

   $$f_n \leq \varphi^{n-1}.$$

[1] *Clearly* identify the *specific* error — vagueness not acceptable here!

(Be careful! There's a slight trap in this question, into which you may fall if you are not careful.)

Solution: We proceed by induction on n, with the base case $n = 1$ trivial ($f_1 = 1 = \varphi^0$).

For the induction step, assume that for some arbitrary $n \geq 1$ we have $f_n \leq \varphi^{n-1}$. We have

$$\begin{aligned} f_{n+1} &= f_n + f_{n-1} \quad \text{(definition)} \\ &\leq \varphi^{n-1} + \varphi^{n-2} \quad \text{(induction hypothesis)} \\ &= \varphi^{n-2}(\varphi + 1) \\ &= \varphi^{n-2}(\varphi^2) \quad \text{(simple algebra)} \\ &= \varphi^n = \varphi^{(n+1)-1}, \end{aligned}$$

and so, by induction on n, $f_n \leq \varphi^{n-1}$ for all n.

Where was the trap? There are two.

1 The induction hypothesis $f_n \leq \varphi^{n-1}$ *does not also* allow us to assume $f_{n-1} \leq \varphi^{n-2}$, but in the above proof we did in fact do that. The fix is to use the principle of *strong* induction, which in the induction step allows us to derive $p(n+1)$ from *all of* $p(1), p(2), \ldots, p(n)$.

2 Even with the above issue dealt with, there is a problem right at the first step of the induction, at $n = 1$. When you write

$$f_2 = f_1 + f + 0$$

and by induction bound $f_1 \leq \varphi^0$, you can't also bound $f_0 \leq \varphi^{-1}$ "by induction" (which is what the solution is doing), because this clause is definitely not part of the statement to be proven. So the above argument has a gap at $n = 1$, that is, in proving $f_2 \leq \varphi$.

One fix is to deal with the f_2 separately; consider it another base case. The assertion to be proven is that $f_2 \leq \varphi$, which is trivial. Then move on to the induction step, assuming $n \geq 2$.

Another fix is to initially observe that the claimed bound also works for $n = 0$, because $f_0 = 0 \leq \varphi^{-1}$, and then proceed with the induction above.

In either case, the fix is to verify *two* base cases before moving on to the induction step — either $n = 1, 2$ or $n = 0, 1$. This often happens when proving things by induction about sequences defined recursively.

(b) Prove that that for $n \geq 1$

$$f_n \geq \varphi^{n-2}.$$

Solution: We proceed by *strong* induction on n. When $n = 1$ the assertion is

$$f_1 \geq \varphi^{-1},$$

which is true since $1 \geq 2/(1 + \sqrt{5})$.

We also verify directly the case $n = 2$, which asserts (correctly)

$$1 \geq 1.$$

Notice here that we *cannot* verify $n = 0$, and use induction for $n = 2$, because here the $n = 0$ case ($0 \geq \varphi^{-2}$) is *false*.

Now we proceed to the induction step. Assume that for some arbitrary $n \geq 2$, we know that assertion is true for all k, $1 \leq k \leq n$. We have

$$\begin{aligned} f_{n+1} &= f_n + f_{n-1} \quad \text{(definition)} \\ &\geq \varphi^{n-2} + \varphi^{n-3} \quad \text{(induction hypothesis)} \\ &= \varphi^{n-3}(\varphi + 1) \\ &= \varphi^{n-3}(\varphi^2) \quad \text{(simple algebra)} \\ &= \varphi^{n-1} = \varphi^{(n+1)-2}, \end{aligned}$$

and so, by induction on n, $f_n \geq \varphi^{n-2}$ for all $n \geq 1$.

Note: These two parts together show that f_n grows roughly at the same rate as φ^n; specifically, for all $n \geq 1$

$$0.3819 \approx \frac{1}{\varphi^2} \leq \frac{f_n}{\varphi^n} \leq \frac{1}{\varphi} \approx 0.6180.$$

It's possible to be more precise, and show that for all large n

$$\frac{f_n}{\varphi^n} \approx \frac{1}{\sqrt{5}} \approx 0.4472.$$

(And it's possible to be *much* more precise, and give an exact formula for f_n in terms of φ).

7. Prove that for all natural numbers n, the expression

$$2 \times 7^n + 3 \times 5^n - 5$$

is divisible by 24. (It will be helpful to know that if a divides b, and a divides c, then a divides any linear combination of b and c; that is, a divides $mb + nc$ for every pair of integers m, n).

Solution: We proceed by (an *outer*) induction on n, with the base case $n = 1$ straightforward.

For the induction step, suppose that 24 divides $2 \times 7^n + 3 \times 5^n - 5$, and consider the number

$$2 \times 7^{n+1} + 3 \times 5^{n+1} - 5.$$

We wish to show that this number is divisible by 24. Now

$$\begin{aligned} 2 \times 7^{n+1} + 3 \times 5^{n+1} - 5 &= 14 \times 7^n + 15 \times 5^n - 5 \\ &= 5\,(2 \times 7^n + 3 \times 5^n - 5) + 4 \times 7^n + 20. \end{aligned}$$

By induction we know that 24 divides $5\,(2\times 7^n + 3\times 5^n - 5)$, show we need to show that also 24 divides $4\times 7^n + 20$, which is the same as showing that 6 divides $7^n + 5$. We prove *this* by (an *inner*) induction on n, with the base case $n = 1$ easy. For the induction step, note that

$$7^{n+1} + 5 = 7(7^n + 5) - 30.$$

By the (inner) induction hypothesis, 6 divides $7(7^n + 5)$, and since also 6 divides 30 we get that 6 divides $7^{n+1} + 5$, completing the (inner) induction, and so also completing the (outer) induction.

In this example we had an induction within an induction, which of course is perfectly OK — the "inner" induction makes no reference to the "outer" one.

8. Prove the generalized triangle inequality: for all natural numbers n, if $x_1, x_2, \ldots, x_n$ are real numbers, then

$$|x_1 + x_2 + \cdots + x_n| \le |x_1| + |x_2| + \cdots + |x_n|.$$

Solution: We proceed by induction on n. The base case $n = 1$ is trivial. The case $n = 2$ is the ordinary triangle inequality, which we have proven in class.

For $n \ge 2$, assume $|x_1 + x_2 + \cdots + x_n| \le |x_1| + |x_2| + \cdots + |x_n|$ for all real numbers $x_1, x_2, \ldots, x_n$, and let $x_1, x_2, \ldots, x_n, x_{n+1}$ be any collection of $n + 1$ real numbers. We have

$$\begin{aligned} |x_1 + x_2 + \cdots + x_n + x_{n+1}| &= |x_1 + x_2 + \cdots + x_{n-1} + (x_n + x_{n+1})| \\ &\le |x_1| + |x_2| + \cdots + |x_{n-1}| + |x_n + x_{n+1}| \quad \text{(induction hypothesis)} \\ &= |x_1| + |x_2| + \cdots + |x_n| + |x_{n+1}| \quad \text{(ordinary triangle inequality)}, \end{aligned}$$

and we are done by induction.

Alternatively, we have

$$\begin{aligned} |x_1 + x_2 + \cdots + x_n + x_{n+1}| &= |(x_1 + x_2 + \cdots + x_n) + x_{n+1}| \\ &\le |x_1 + x_2 + \cdots + x_n| + |x_{n+1}| \quad \text{(ordinary triangle inequality)} \\ &= |x_1| + |x_2| + \cdots + |x_{n-1}| + |x_n| + |x_{n+1}| \quad \text{(induction hypothesis)}, \end{aligned}$$

and again we are done by induction.

9. For whole numbers $n \ge \ell \ge 0$ let

$$f(n, \ell) = \sum_{k=0}^{\ell} (-1)^k \binom{n}{k}$$

(so $f(n, \ell)$ is the alternating sum of the entries along the nth row of Pascal's triangle, up to and including the term $\binom{n}{\ell}$). For example

- $f(0, 0) = (-1)^0 \binom{0}{0} = 1$,

- $f(1,0) = (-1)^0\binom{1}{0} = 1$,
- $f(1,1) = (-1)^0\binom{1}{0} + (-1)^1\binom{1}{1} = 0$,
- $f(2,0) = (-1)^0\binom{2}{0} = 1$,
- $f(2,1) = (-1)^0\binom{2}{0} + (-1)^1\binom{2}{1} = -1$,
- $f(2,2) = (-1)^0\binom{2}{0} + (-1)^1\binom{2}{1} + (-1)^2\binom{2}{2} = 0$, and
- $f(5,3) = (-1)^0\binom{5}{0} + (-1)^1\binom{5}{1} + (-1)^2\binom{5}{2} + (-1)^3\binom{5}{3} = -4$.

By computing $f(n,\ell)$ for a bunch more small values of n and ℓ (by hand, or by computer), conjecture a simple formula for $f(n,\ell)$ and prove that the formula is correct.

Solution: After calculating a bunch of values, it appears that

$$f(n,\ell) = \begin{cases} 1 & \text{if } n = \ell = 0 \\ 0 & \text{if } n = \ell > 0 \\ (-1)^\ell\binom{n-1}{\ell} & \text{otherwise.} \end{cases}$$

To prove this, first note that the case $n = \ell = 0$ is an easy calculation. For $n = \ell > 0$, we use the binomial theorem:

$$0 = 0^n = ((-1)+1)^n = \sum_{k=0}^{n}\binom{n}{k}(-1)^k(1)^{n-k} = f(n,n).$$

That leaves the remaining cases, where $n > \ell$ and $n > 0$. For each fixed n, we prove by induction on ℓ the predicate $p(\ell)$: $\sum_{k=0}^{\ell}(-1)^k\binom{n}{k} = (-1)^\ell\binom{n-1}{\ell}$. The base case $\ell = 0$ is easy: both sides evaluate to 1. For the induction step, suppose $p(\ell)$ is true. We have

$$\begin{aligned}
\sum_{k=0}^{\ell+1}(-1)^k\binom{n}{k} &= \left(\sum_{k=0}^{\ell}(-1)^k\binom{n}{k}\right) + (-1)^{\ell+1}\binom{n}{\ell+1} \\
&= (-1)^\ell\binom{n-1}{\ell} + (-1)^{\ell+1}\binom{n}{\ell+1} \quad \text{(induction hypothesis)} \\
&= (-1)^\ell\left(\binom{n-1}{\ell} - \binom{n}{\ell+1}\right) \\
&= (-1)^\ell\left(-\binom{n-1}{\ell+1}\right) \quad \text{(Pascal's identity)} \\
&= (-1)^{\ell+1}\binom{n-1}{\ell+1},
\end{aligned}$$

and we are done by induction.

But wait, what is going on here? We are using induction to prove $p(\ell)$ for *finitely* many values of ℓ, namely $\ell = 0, 1, \ldots, n-1$, whereas induction (as we have seen it) is used to prove things for *infinitely* many values of the parameter! And here, the predicate we are proving doesn't even make sense for $\ell \geq n$ (since it involves a binomial coefficient where the number downstairs is bigger than the number upstairs, and we haven't defined such a monster).

In fact, induction is often used to prove statements like this for finite ranges of a parameter, and it's all perfectly legitimate. What's really happening is this: the predicate we are proving, for all $\ell \geq 0$, is

$$p(\ell) : 0 \leq \ell < n \text{ implies } \sum_{k=0}^{\ell}(-1)^k\binom{n}{k} = (-1)^\ell\binom{n-1}{\ell}$$

or, equivalently,

$$p(\ell) : \ell \geq n \text{ or } \sum_{k=0}^{\ell}(-1)^k\binom{n}{k} = (-1)^\ell\binom{n-1}{\ell}$$

The induction step that we have described (showing $p(\ell)$ implies $p(\ell+1)$) works perfectly in the case $0 \leq \ell < n$. If $\ell \geq n$, then we have a different, trivial, induction step: $p(\ell)$ implies $p(\ell+1)$ in this case trivially, because both the premise and the hypothesis in this case are just plain true, so the implication is true.

10. In class we saw that the general associative law — no matter how parentheses are placed around the expression $a_1 + a_2 + \ldots + a_n$, the sum is still the same — follows from the associativity axiom.

    Show that the general commutative law — no matter what order $a_1, a_2, \ldots, a_n$ are added in, the sum is still the same — follows from the commutativity axiom $a+b = b+a$. You may assume the general associative law.

    (As a specific clarifying example, the case $n = 3$ of the general commutative law says that $a+b+c$, $a+c+b$, $b+a+c$, $b+c+a$, $c+a+b$ and $c+b+a$ are all the same.)

    Solution: Some notation will be useful. Write GCA(n) for the predicate "for all real numbers $a_1, \ldots, a_n$, no matter what order $a_1, a_2, \ldots, a_n$ are added in, the sum is still the same".

    We will prove $(\forall n)$GCA(n) by *complete* induction on n. The base case GCA(1) requires nothing to prove, and the base case GCA(2) is the commutativity axiom. Now suppose that for some n, we know GCA(k) for all k in the range $1 \leq k \leq n$. We will show GCA($n+1$), by showing that for all real numbers $a_1, \ldots, a_n, a_{n+1}$, no matter what order $a_1, a_2, \ldots, a_n, a_{n+1}$ are added in, the sum is always the same as the *natural* sum $N(a_1, \ldots, a_{n+1}) = a_1 + a_2 + \cdots + a_{n+1}$.

    So, let $n+1$ real numbers $a_1, \ldots, a_{n+1}$ be given. Let S be a summing of the numbers in some order. We precede by cases.

    Case 1: $S = \text{SOMETHING} + a_{n+1}$. In this case, by GCA($n$) we have $\text{SOMETHING} = N(a_1, \ldots, a_n) = a_1 + \cdots + a_n$ so $S = a_1 + \cdots + a_n + a_{n+1} = N(a_1, \ldots, a_{n+1})$, as required.

    Case 2 $S = \text{SOMETHING}_1 + a_{n+1} + \text{SOMETHING}_2$, where SOMETHING_1 might be empty, but SOMETHING_2 is not empty (i.e., is genuinely the sum of some of the a_i). In this case we can use commutativity (GCA(2)) to write $S = \text{SOMETHING}_1 + \text{SOMETHING}_2 + a_{n+1}$. We are now back is Case 1, and the argument given there shows $S = a_1 + \cdots + a_n + a_{n+1} = N(a_1, \ldots, a_{n+1})$, as required.

    So, by strong induction, we are done.

(But really this was only strong induction because we used GCA(2) as well as GCA(n) to deduce GCA($n+1$). Had we appealed to "commutativity" rather than GCA(2), we could have called this a proof by ordinary induction.)