Math 10850, Honors Calculus 1

Homework 5

Solutions

1. Let $f(x) = 1/(1+x)$.

   (a) What is $f(f(x))$? And what is the domain of this new function?

   Solution:

   $$f(f(x)) = \frac{1}{1+f(x)} = \frac{1}{1+\frac{1}{1+x}} = \frac{1+x}{2+x} \quad \text{(this last equality valid as long as } x \neq -1\text{)}.$$

   The domain is $\{x \in \mathbb{R} : x \neq -1, -2\}$. bf NB: -1 is **not** in the domain, even though the final expression makes perfect sense at $x = -1$. The domain of the composition $g \circ h$ is the set of all domain elements of the first function (h, or in this case f), whose outputs are in the domain of the second function (g, or in this case f); and -1 is **not** in the domain of the first function.

   (b) What is $f(cx)$, where c is some fixed real number? And what is the domain of this new function?

   Solution: This new function is really a composition: $f \circ m_c$ where m_c is the "multiply by c" function, whose domain is all reals.

   $$f(cx) = \frac{1}{1+cx}.$$

   If $c = 0$, the domain is all reals. If $c \neq 0$ the domain is $\mathbb{R} \setminus \{-1/c\}$ (or $(-\infty, -1/c) \cup (-1/c, \infty)$ or $\{x \in \mathbb{R} : x \neq -1/c\}$).

   (c) For which real numbers c, is there a number x such that $f(cx) = f(x)$?

   Solution: Given c, we want to find an x such that

   $$\frac{1}{1+cx} = \frac{1}{1+x}$$

   with $x \neq -1, -1/c$. This is equivalent to $cx = x$ or $(c-1)x = 0$, which can be solved for **all** c by taking $x = 0$.
   Solution summary: all $c \in \mathbb{R}$.

(d) For which numbers c is it true that $f(cx) = f(x)$ for (at least) two different numbers x?

Solution: As in the last part, for given $c \in \mathbb{R}$ we are trying to solve $(c-1)x = 0$ with $x \neq -1, -1/c$. If $c \neq 1$ then $c - 1 \neq 0$ so the *only* solution to $(c-1)x = 0$ is $x = 0$. If $c = 1$ then the equation becomes $0.x = 0$, which has *infinitely many* solutions (and so certainly has two different solutions).

Solution summary: only $c = 1$.

2. Find the domain of each of these functions.

(a) $f(x) = \sqrt{1 - \sqrt{1 - x^2}}$.

Solution: In order for the inner $\sqrt{1 - x^2}$ to make sense, we need $1 - x^2 \geq 0$, or $x \in [-1, 1]$. For all such x, we have $0 \leq \sqrt{1 - x^2} \leq 1$, so $1 \geq 1 - \sqrt{1 - x^2} \geq 0$. This means that for all $x \in [-1, 1]$, the expression $\sqrt{1 - \sqrt{1 - x^2}}$ makes sense, and so the domain of this last function is $[-1, 1]$.

(b) $f(x) = 1/(x - 1) + 1/(x - 2)$.

Solution: The domain is $\mathbb{R} \setminus \{1, 2\}$.

(c) $f(x) = \sqrt{1 - x^2} + \sqrt{x^2 - 1}$.

Solution: In order for $\sqrt{1 - x^2}$ to make sense, we need $1 - x^2 \geq 0$, or $x \in [-1, 1]$. In order for $\sqrt{x^2 - 1}$ to make sense, we need $x^2 - 1 \geq 0$, or $x \in (-\infty, -1] \cup [1, \infty)$. In order for both to make sense simultaneously, we need x to be in the intersection of these two sets, that is, we need $x = 1$ or -1. So the domain is $\{-1, 1\}$.

3. A function f is said to be *even* if $f(x) = f(-x)$ for all x, and *odd* if $f(x) = -f(-x)$ for all x (so $f(x) = |x|$ is even and $f(x) = x^3$ is odd, for example).

(a) Determine whether $f + g$ is even, odd, or not necessarily either in the four cases obtained by choosing f even or odd, and g even or odd. (**Note**: here, if you think that $f + g$ is (say) even, then you should show why $(f + g)(-x) = (f + g)(x)$ follows from whatever properties of f, g you are assuming. If you think that it's not possible to say definitely whether $f + g$ is odd or even, you should give explicit examples. Typically, it will suffice to produce a *single* example of a pair f, g with $f + g$ *neither* even *nor* odd — witnesses by an x with $(f + g)(-x) \neq (f + g)(x)$ and $(f + g)(-x) \neq -(f + g)(x)$.)

Solution:

- f, g both even: $(f + g)(-x) = f(-x) + g(-x) = f(x) + g(x) = (f + g)(x)$, so $f + g$ is even in this case.
- f even, g odd: $(f + g)(-x) = f(-x) + g(-x) = f(x) - g(x) = (f - g)(x)$. This suggests that in general $f + g$ is neither even nor odd in this case. For example, take $f(x) = x^2$ and $g(x) = x$, so $(f + g) = x^2 + x$. This is neither even nor odd ($f(-1) = 0$, which is equal to neither $f(1)$ nor $-f(1)$).

- f odd, g even: Same as last case, by symmetry; in general $f+g$ is neither even nor odd in this case.
- f, g both odd: $(f+g)(-x) = f(-x) + g(-x) = -f(x) - g(x) = -(f+g)(x)$, so $f+g$ is odd in this case.

(b) Do the same for $f \cdot g$.

Solution:

- f, g both even: $(fg)(-x) = f(-x)g(-x) = f(x)g(x) = (fg)(x)$, so fg is even in this case.
- f even, g odd: $(fg)(-x) = f(-x)g(-x) = f(x)(-g(x)) = -(fg)(x)$, so fg is odd in this case.
- f odd, g even: Same as last case, by symmetry; fg is odd in this case.
- f, g both odd: $(fg)(-x) = f(-x)g(-x) = (-f(x))(-g(x)) = (-1)^2(fg)(x) = fg(x)$, so fg is even in this case.

(c) Do the same for $f \circ g$.

Solution:

- f, g both even: $(f \circ g)(-x) = f(g(-x)) = f(g(x)) = (f \circ g)(x)$, so $f \circ g$ is even in this case (notice that we did not use that f is even in this case — if g is even and f is *anything* then $f \circ g$ is even).
- f even, g odd: $(f \circ g)(-x) = f(g(-x)) = f(-g(x)) = f(g(x)) = (f \circ g)(x)$, so $f \circ g$ is even in this case (here we used properties of both f and g).
- f odd, g even: $(f \circ g)(-x) = f(g(-x)) = f(g(x)) = (f \circ g)(x)$, so $f \circ g$ is even in this case (we could have deduced this from our analysis of the case where both are even).
- f, g both odd: $(f \circ g)(-x) = f(g(-x)) = f(-g(x)) = -f(g(x)) = -(f \circ g)(x)$, so $f \circ g$ is odd in this case (here we used properties of both f and g).

(d) Prove that every even function f can be written as $f(x) = g(|x|)$, for *infinitely many* functions g.

Solution: Let c be any real number, and define $g_c : \mathbb{R} \to \mathbb{R}$ as follows:

$$g_c(x) = \begin{cases} f(x) & \text{if } x \geq 0 \\ c & \text{if } x < 0. \end{cases}$$

If $x \geq 0$ then $g_c(|x|) = g_c(x) = f(x)$, while if $x < 0$ then $g_c(|x|) = g_c(-x) = f(-x)$ (note that $-x \geq 0$ in this case), and $f(-x) = f(x)$ (since x is even), so $g_c(|x|) = f(x)$. So for all real x, $g_c(|x|) = f(x)$.

Clearly there are infinitely many (different) functions of the form f_c, since there are infinitely many c.

The key point here is: since $g(|x|)$ only uses the rule for g on positive inputs, we can do whatever we like on negative inputs.

4. For each of the following assertions, either give a proof (if the assertion is true) or a counterexample (if it is false).

   (a) $f \circ (g + h) = f \circ g + f \circ h$.

   Solution: This is **false** in general. Consider, for example, $g = h$ both being the identity function (that sends x to x) and f being the function that sends x to x^2. We have

   $$(f \circ (g + h))(x) = f((g + h)(x)) = f(g(x) + h(x)) = f(x + x) = f(2x) = 4x^2$$

   and

   $$((f \circ g) + (f \circ h))(x) = (f \circ g)(x) + (f \circ h)(x) = f(g(x)) + f(h(x)) = f(x) + g(x) = x^2 + x^2 = 2x^2,$$

   and the two ($4x^2$ and $2x^2$) are not in general equal.

   (b) $(g + h) \circ f = g \circ f + h \circ f$.

   Solution: We have

   $$\begin{aligned} ((g + h) \circ f)(x) &= (g + h)(f(x)) \\ &= g(f(x)) + h(f(x)) \\ &= (g \circ f)(x) + (h \circ f)(x) \\ &= ((g \circ f) + (h \circ f))(x) \end{aligned}$$

   so $(g + h) \circ f = (g \circ f) + (h \circ f)$.

   (c) $1/(f \circ g) = (1/f) \circ g$.

   Solution: We have

   $$\begin{aligned} \left(\frac{1}{f \circ g}\right)(x) &= \frac{1}{(f \circ g)(x)} \\ &= \frac{1}{f(g(x))} \\ &= \left(\frac{1}{f}\right)(g(x)) \\ &= \left(\frac{1}{f} \circ g\right)(x) \end{aligned}$$

   so $1/(f \circ g) = (1/f) \circ g$.

   (d) $1/(f \circ g) = f \circ (1/g)$.

   Solution: This is **false** in general. Consider, for example $f(x) = x + 1$ and $g(x) = x^2$. We have

   $$(1/(f \circ g))(x) = 1/((f \circ g)(x)) = 1/(f(g(x))) = 1/(f(x^2)) = 1/(x^2 + 1)$$

and

$$(f \circ (1/g))(x) = f((1/g)(x)) = f(1/g(x)) = f(1/x^2) = (1/x^2) + 1,$$

and $1/(x^2+1)$ is not equal to $(1/x^2)+1$ for all x (i.e., $x = 2$).

5. Indicate on the real number line the set of all x satisfying the following conditions. Also express each set in interval notation (possibly using $\cup$).

   (a) $|x^2 - 1| < 1/2$.

   Solution: $|x^2 - 1| < 1/2$ if and only if $-1/2 < x^2 - 1 < 1/2$ if and only if $1/2 < x^2 < 3/2$ if and only if EITHER $-\sqrt{3/2} < x < -\sqrt{1/2}$ (case of x negative) OR $\sqrt{1/2} < x < \sqrt{3/2}$ (case of x positive), interval(s) $(-\sqrt{3/2}, -\sqrt{1/2}) \cup (\sqrt{1/2}, \sqrt{3/2})$. (Number line not shown.)

   (b) $1/(1+x^2) \le a$ (the answer may depend on a, so you may have to consider cases).

   Solution: We have $1 + x^2 > 0$ always so $1/(1+x^2) > 0$ always, so if $a \le 0$ there is no solution (we are asking for x to satisfy $0 \ge a \ge 1/(1+x^2) > 0$).
   For $a > 0$ we have $1/(1+x^2) \le a$ if and only if $(1-a)/a \le x^2$. If $a \ge 1$ then $(1-a)/a \le 0$, and so the fact that $x^2 \ge 0$ for all x implies that automatically $(1-a)/a \le x^2$, for all x (interval $(-\infty, \infty)$). If $0 < a < 1$ then $(1-a)/a \le x^2$ is equivalent to EITHER $x \le -\sqrt{(1-a)/a}$ (case of x negative) OR $x \ge \sqrt{(1-a)/a}$ (case of x positive), interval $(-\infty, -\sqrt{(1-a)/a}] \cup [\sqrt{(1-a)/a}, \infty)$ (note the use of square brackets!).
   Summary: Case $a \le 0$, solution set empty. Case $a \ge 1$, solution set $(-\infty, \infty)$. Case $0 < a < 1$, solution set $(-\infty, -\sqrt{(1-a)/a}] \cup [\sqrt{(1-a)/a}, \infty)$.
   (Number line not shown.)

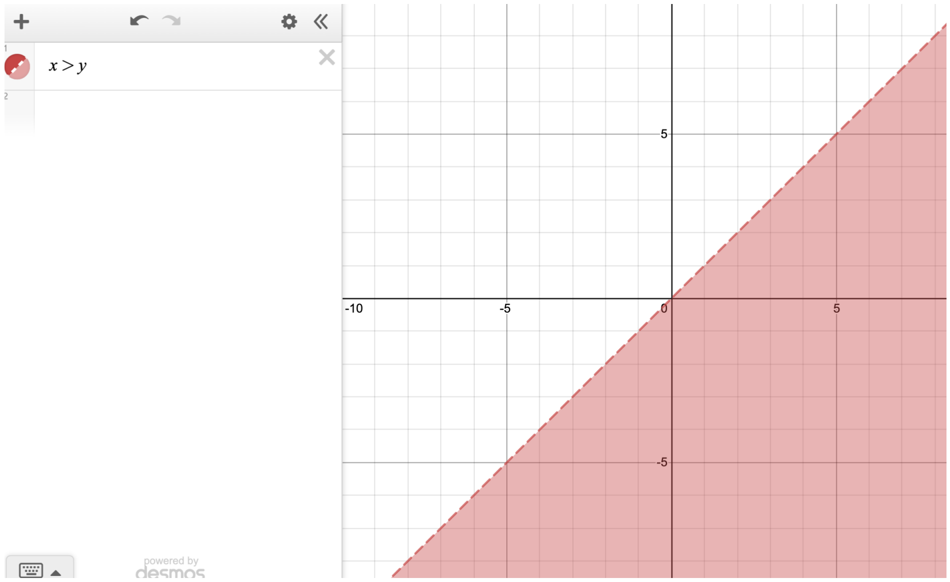
In the following two questions, I'm asking you to draw graphs/regions of the plane. **Please** draw your axes using a straightedge; label both axes, and include a scale on both axes! (Otherwise, your graph is not fully interpretable). Give your graphs a little space — don't try to scrunch two of them side-by-side on the page. When an endpoint of a continuous[1] piece of your graph is not part of the graph, indicate this by a hollow circle. When it is part of the graph, indicate this by a solid circle.

6. Sketch in the coordinate plane the set of points (x, y) satisfying:

   (a) the inequality $x > y$.

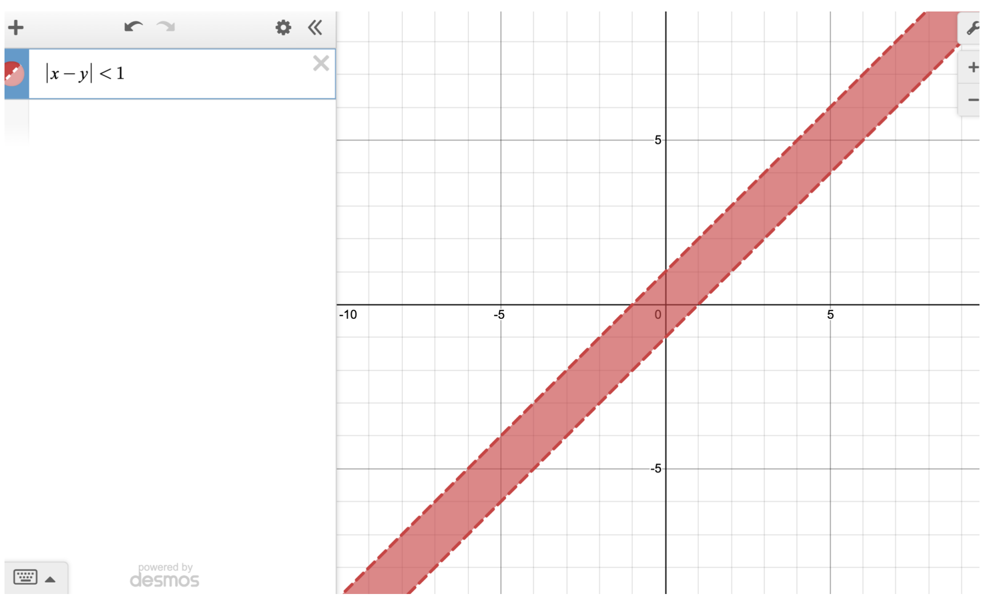
   Solution: The red shaded region in the graph below. Note that the line $x = y$ is not part of the region; this is indicated with a dashed line. Since Desmos doesn't label axes, I'll say here (and this applies to the rest of the graphs in this solution set) that the y-axis is oriented vertically (with up being positive) and the y-axis horizontally (with to the right being positive).

[1] Continuous? What does that word mean? I have no idea. But, I'll learn about it in the next week or so, and tell you in class...



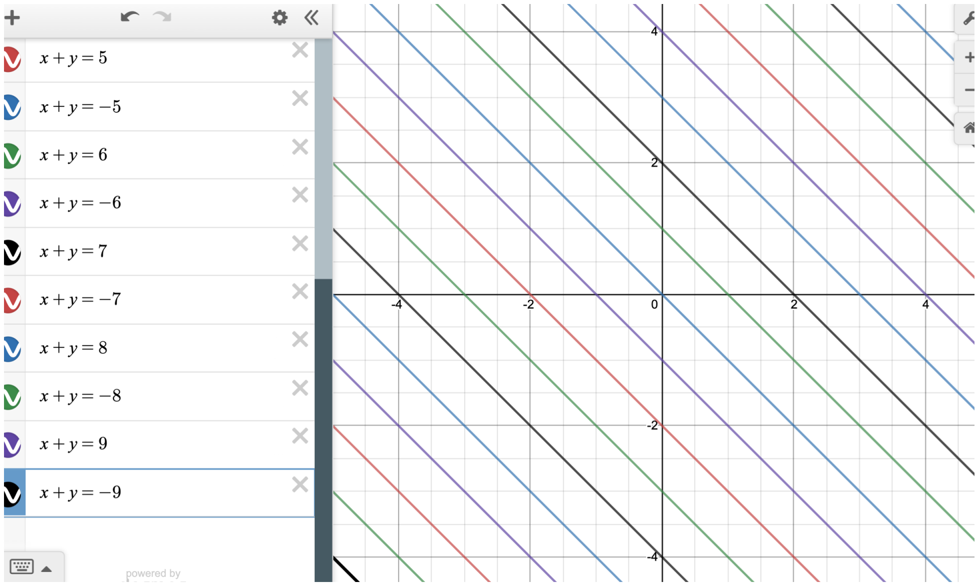
(b) the inequality $|x - y| < 1$.

Solution: $|x - y| < 1$ is equivalent to $-1 < x - y < 1$ which is equivalent to BOTH $-1 < x - y$ AND $x - y < 1$, which is equivalent to BOTH $y < x + 1$ AND $y > x - 1$. We draw the two (parallel) lines $y = x + 1$, $y = x - 1$, for each line check which side of the line satisfies the relevant inequality, then take intersection of these two sides. The red shaded region in the graph below.



(c) the condition $x + y \in \mathbb{Z}$.

Solution: This is the union of all the lines $x + y = z$, where z runs over all integers.

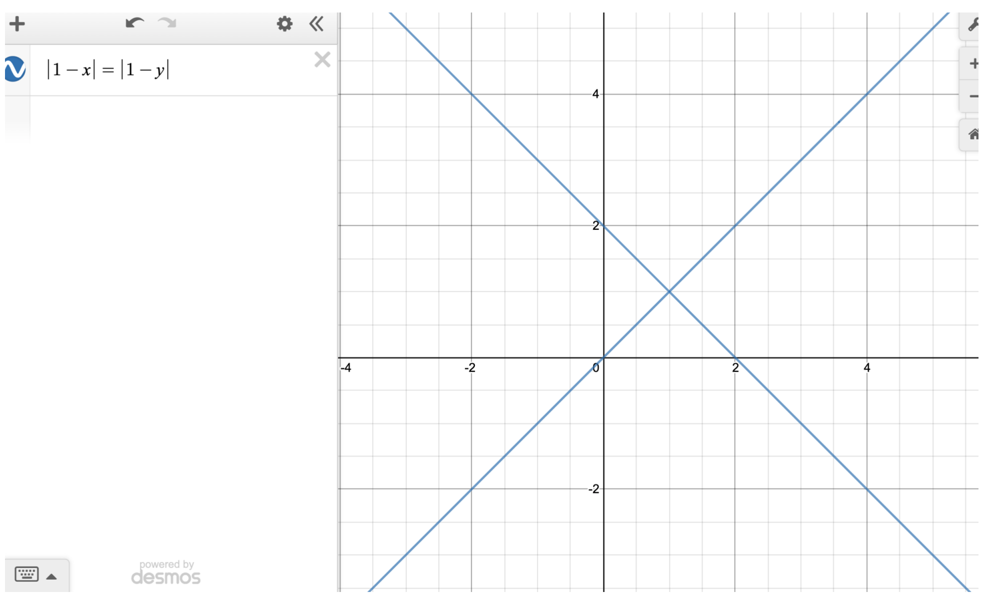


(d) $|1 - x| = |y - 1|$

Solution: We treat cases, according to where x is relative to 1 and where y is relative to 1.

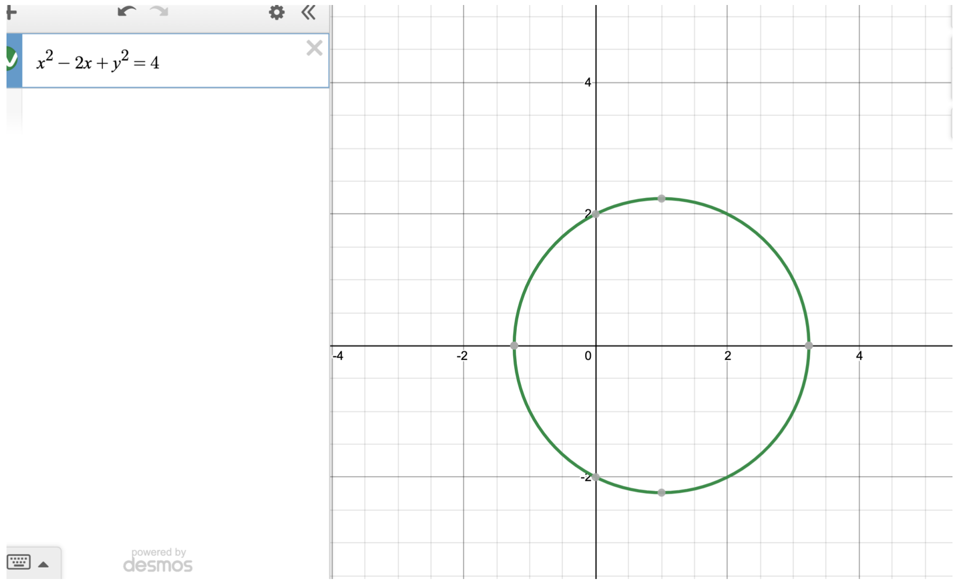
- **Case 1** $x \geq 1$, $y \geq 1$: Equation becomes $x - 1 = y - 1$ or $x = y$.
- **Case 2** $x \geq 1$, $y \leq 1$: Equation becomes $x - 1 = 1 - y$ or $x + y = 2$.
- **Case 3** $x \leq 1$, $y \geq 1$: Equation becomes $1 - x = y - 1$ or $x + y = 2$.
- **Case 4** $x \leq 1$, $y \leq 1$: Equation becomes $1 - x = 1 - y$ or $x = y$.

Putting together, we get two perpendicular lines intersecting at $(1, 1)$.



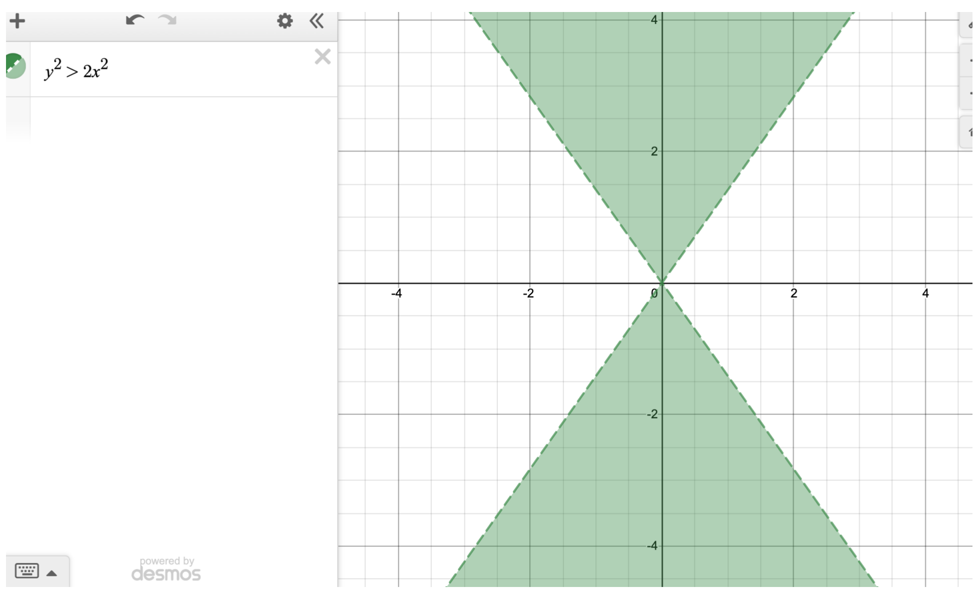
(e) $x^2 - 2x + y^2 = 4$.

Solution: $x^2 - 2x + y^2 = 4$ is the same as $(x-1)^2 + (y-0)^2 = (\sqrt{5})^2$, so the set of points determined is a circle, radius $\sqrt{5}$, centered at $(1, 0)$.



(f) $y^2 > 2x^2$.

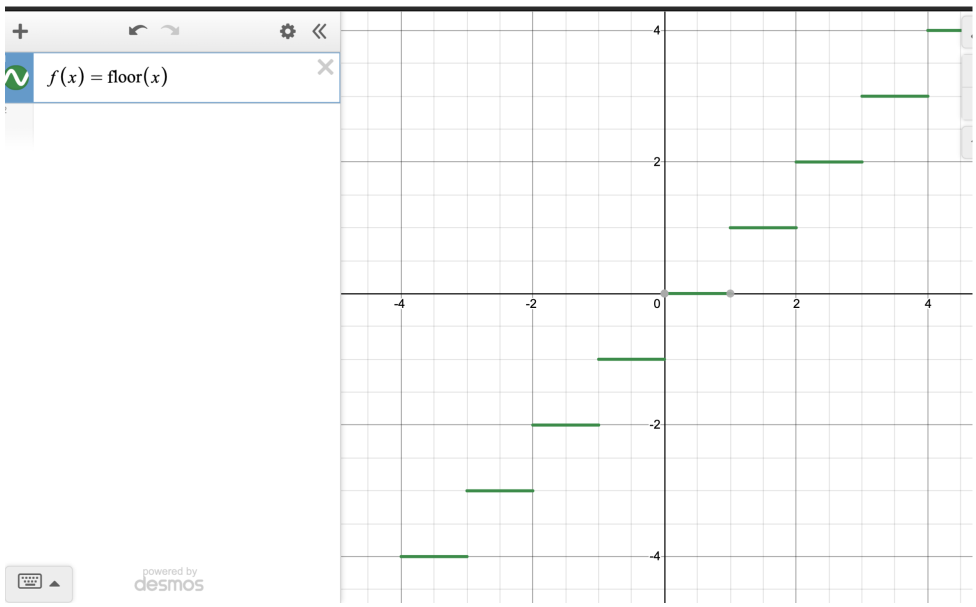
Solution: $y^2 = 2x^2$ determines two straight lines through the origin, $y = \sqrt{2}x$ (slope $\sqrt{2}$) and $y = -\sqrt{2}x$ (slope $-\sqrt{2}$). These two lines partition the plane into 4 connected regions, two of which satisfy $y^2 > 2x^2$ (found by tested points in each of the four regions) and two of which do not. The two lines themselves do not form part of the region.
The green shaded region in the graph below.



7. The symbol $[x]$ denotes the largest integer which is $\leq x$; it's called the *integer part* of x. So, for example, $[2.1] = 2$, $[2] = 2$, $[-0.9] = -1$ and $[-1] = -1$.. Draw the graph of the following functions:

(a) $f(x) = [x]$.

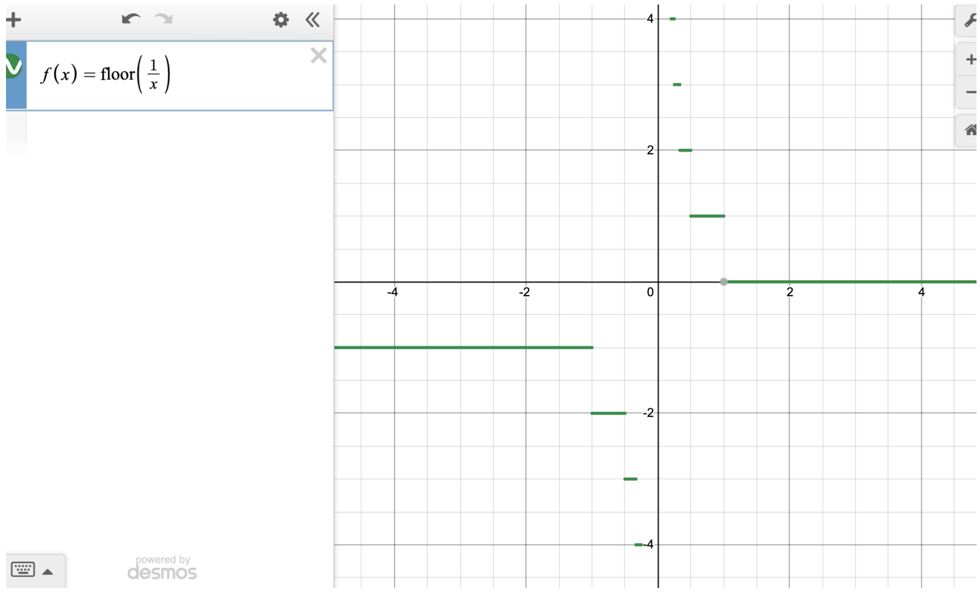
Solution: For k an integer, on the interval from k to $k+1$, the integer part takes the value k, **except** at $k+1$, where it jumps to $k+1$. So the graph will be a collection of line segments of length 1, stepping up by one each time one moves one unit to the right, with the intervals being closed on the left ($[k] = k$) and open on the right ($[k+1] \neq k$). Note that it is traditional to use a solid dot for a closed endpoint of an interval, and a hollow dot for an open endpoint; unfortunately I cannot get Desmos to do this :(. Note also that Desmos uses "floor" for this function, and this language is very common.



(b) $f(x) = [1/x]$.

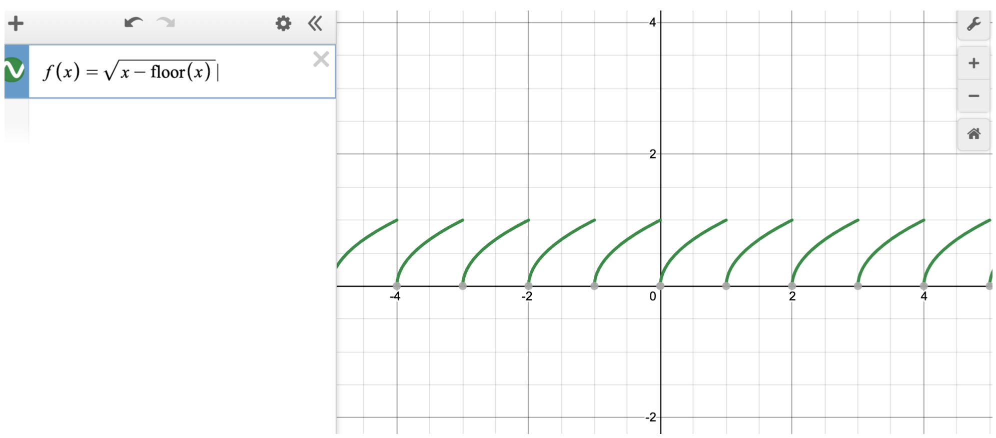
Solution: We start by studying what happens when x is positive. When $x > 1$, $0 < 1/x < 1$ so $[1/x] = 0$. When $1/2 < x \leq 1$, $1 \leq 1/x < 2$, so $[1/x] = 1$. When $1/3 < x \leq 1/2$, $2 \leq 1/x < 3$, so $[1/x] = 2$. This pattern continues — as x gets closer to 1, $[1/x]$ grows larger, and its intervals of being constant grow ever shorter. Those intervals, which are of the form $(1/(k+1), 1/k]$, $k = 0, 1, 2, \ldots$ ($1/0$ locally interpreted to mean ∞) are open on the left and closed on the right, and along the particular interval $(1/(k+1), 1/k]$ the value taken is k.

On the negative side: on the interval $(-1/k, -1/(k+1)]$, $k = 0, 1, 2, \ldots$ ($-1/0$ locally interpreted to mean $-\infty$) we have $-1/k < x \leq -1/(k+1)$, or $-(k+1) \leq 1/x < -k$, or $[1/x] = -(k+1)$. Notice the subtle difference between positive and negative sides: although on both sides the intervals are open on left, closed on right, on the negative side, the function stats at -1 and drops, while on the positive side, it starts at 0 and rises.



(c) $f(x) = \sqrt{x - [x]}$.

Solution: For k *any* integer, as x goes from k to $k+1$ (including k, not including $k+1$), $x - [x]$ goes from 0 to 1 linearly (including 0, not including 1).
So on $[0, 1)$ the graph looks like the graph of $f(x) = \sqrt{x}$, and then the rest of the graph is a sequence of translates of this portion. The continuous intervals below should have closed circles on the left ends, open circles on the right.



8. Find the domains of each of the following functions:

(a) $f(x) = \sqrt{1 - x} + \sqrt{2 - x}$

Solution: For $\sqrt{1 - x}$ to make sense we require $x \leq 1$, and $\sqrt{2 - x}$ to make sense we require $x \leq 2$; for both of these things to happen we require $x \leq 1$, so $\text{Domain}(f) = \{x : x \leq 1\} = (-\infty, 1]$.

(b) $g(x) = 1/\sqrt{x^2 - 5x + 6}$

Solution: We require $x^2 - 5x + 6 > 0$. Since $x^2 - 5x + 6 = (x-2)(x-3)$, this is equivalent to $x > 3$ or $x < 2$. So Domain$(f) = \{x : x > 3 \text{ or } x < 2\} = (-\infty, 2) \cup (3, \infty)$.

(c) $h_2 = h_1 \circ h_1$ where $h_1 = -1/x$ for $x > 0$ and undefined otherwise.

Solution: We require $x > 0$ for $h_1(x)$ to make sense. For $h_1(h_1(x))$ to make sense, we require $h_1(x) > 0$. But $h_1(x) < 0$ for all $x > 0$, so there is no $x > 0$ for which $h_1(h_1(x))$ makes sense, i.e. for which $h_1(x)$ is in the domain of h_1. So the domain of f is *empty*.

9. A *parabola* is the set of points in the plane with the following characteristic property: it is the set of points (x, y) such that the distance from (x, y) to a fixed point (a, b) is equal to the distance[2] from (x, y) to a fixed line L (that does not pass through (a, b)). Succinctly, a parabola is the set of points equidistant from a fixed point and a fixed line.

A parabola is not always the graph of some function, but when it is, it is the graph of a quadratic function; and conversely, the graph of a quadratic function is a parabola. This question (essentially) asks you to prove this.

(a) Let L be the horizontal line $y = c$ (c some real constant) and let P be the point (a, b), with $b \neq c$. Prove that the parabola determined by L and P is a set of points of the form $(x, rx^2 + sx + t)$ for some real constants r, s, t (i.e., the parabola is the graph of a specific quadratic function).

Solution: A point (x, y) is on the parabola iff

$$\sqrt{(x-a)^2 + (y-b)^2} = |y - c|$$

which is true iff

$$(x-a)^2 + (y-b)^2 = (|y-c|)^2$$

(we can go from $\sqrt{(x-a)^2 + (y-b)^2} = |y-c|$ to $(x-a)^2 + (y-b)^2 = (|y-c|)^2$ by squaring; to go the other way, note that $(x-a)^2 + (y-b)^2 = (|y-c|)^2$ implies $\sqrt{(x-a)^2 + (y-b)^2} = \pm|y-c|$; but since both $\sqrt{(x-a)^2 + (y-b)^2}$ and $|y-c|$ are positive, it must be that $\sqrt{(x-a)^2 + (y-b)^2} = |y-c|$).
The equation $(x-a)^2 + (y-b)^2 = (|y-c|)^2$ holds iff $(x-a)^2 + (y-b)^2 = (y-c)^2$ (since $(|y-c|)^2 = (y-c)^2$), which holds iff

$$x^2 - 2ax + a^2 + y^2 - 2by + b^2 = y^2 - 2cy + c^2,$$

which holds iff

$$2(b-c)y = x^2 - 2ax + a^2 + b^2 - c^2,$$

[2] The distance from a point P to a line L is defined to be the distance from P to that point on L that is closest to P. It is the distance from P to L along that line through P that is perpendicular to P. If L the horizontal line $y = c$ (c some real constant) and P is the point (a, b), then the distance from P to L is easy to compute: it is $|b - c|$.

which, since $b \neq c$, holds iff

$$y = \frac{x^2}{2(b-c)} - \frac{ax}{b-c} + \frac{a^2+b^2-c^2}{2(b-c)}.$$

So in summary, the parabola determined by L and P is the graph of the quadratic function $f(x) = rx^2 + sx + t$ where

$$r = \frac{1}{2(b-c)}, \quad s = -\frac{a}{b-c}, \quad t = \frac{a^2+b^2-c^2}{2(b-c)}.$$

(b) Let $f(x) = ax^2 + bx + c$ be a quadratic function, with $a > 0$. Find a horizontal line L and a point P, not on L, such that the parabola determined by L and P is exactly the graph of f.

Solution: Let L be the line $y = p$, and let P be the point (v, w) $(w \neq p)$. By the work of the previous part, the parabola determined by L and P is exactly the graph of the function g given by

$$g(x) = \frac{x^2}{2(w-p)} - \frac{vx}{w-p} + \frac{v^2+w^2-p^2}{2(w-p)}.$$

So we need to choose p, v, w $(w \neq p)$ to solve

$$a = \frac{1}{2(w-p)}, \quad b = \frac{v}{w-p}, \quad c = \frac{v^2+w^2-p^2}{2(w-p)}.$$

The first equation tells us

$$w - p = \frac{1}{2a}$$

(note this is valid since $a > 0$). The second equation tells us what v must be: it must be

$$v = \frac{b}{2a}.$$

From the third equation we learn

$$\frac{c}{a} - \frac{b^2}{4a^2} = w^2 - p^2 = (w-p)(w+p),$$

and again using $w - p = 1/2a$ this leads to

$$2c - \frac{b^2}{2a} = w + p.$$

So we have two equations:

$$w + p = 2c - \frac{b^2}{2a}, \quad w - p = \frac{1}{2a}$$

in two unknowns w, p, which we can solve:

$$w = \frac{2c - \frac{b^2}{2a} + \frac{1}{2a}}{2}, \quad p = \frac{2c - \frac{b^2}{2a} - \frac{1}{2a}}{2}.$$

We are done, as we have explicitly identified v, w, p in terms of a, b, c.