Math 10850, Honors Calculus 1

Homework 6

Due in class Friday October 18

General and specific notes on the homework

All the notes from homework 1 still apply!

Reading for this homework

This homework covers limits — section 6 of the class notes, chapter 5 of Spivak.

Assignment

1. In each of the following cases, determine the limit L for the given a, and prove that it is indeed the limit by finding, for each $\varepsilon > 0$, a δ (probably depending on ε) such that $|f(x) - L| < \varepsilon$ for all x satisfying $0 < |x - a| < \delta$.

   (a) $f(x) = 100/x$, $a = 1$.

   (b) $f(x) = x^4 + 1/x$, arbitrary $a > 0$.

2. Calculate the following limits, *not* directly from the definition, but instead using the various theorems we have proven about limits.

   (a) $\lim_{x\to 2} \frac{x^3-8}{x-2}$.

   (b) $\lim_{x\to y} \frac{x^n-y^n}{x-y}$.

   (c) $\lim_{h\to 0} \frac{\sqrt{a+h}-\sqrt{a}}{h}$.

3. For this question, the usual rules apply: if it is your understanding that a certain phenomenon holds in general, then you should provide a proof/justification that that is the case; if it does not hold in general, a single explicit counterexample is enough.

   (a) If $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ both *do not exist*, can $\lim_{x\to a}(f(x) + g(x))$ exist?

   (b) If $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ both *do not exist*, can $\lim_{x\to a} f(x)g(x)$ exist?

   (c) If $\lim_{x\to a} f(x)$ and $\lim_{x\to a}(f(x) + g(x))$ both exist, must $\lim_{x\to a} g(x)$ exist?

   (d) If $\lim_{x\to a} f(x)$ exists and $\lim_{x\to a} g(x)$ does not exist, can $\lim_{x\to a}(f(x) + g(x))$ exist?

   (e) If $\lim_{x\to a} f(x)$ exists and $\lim_{x\to a} f(x)g(x)$ exists, does it follow that $\lim_{x\to a} g(x)$ exists?

4. (a) Prove that $\lim_{x\to 0} f(x) = \lim_{x\to 0} f(x^3)$.
   (b) Give an example where $\lim_{x\to 0} f(x^2)$ exists, but $\lim_{x\to 0} f(x)$ doesn't.

5. Let f, g, h be three functions, and let a be some real number. Suppose that there is some number $\Delta > 0$ such that on the interval $(\Delta - a, \Delta + a)$ it holds that $f(x) \le g(x) \le h(x)$ (except possibly at a, which might or might not be in the domains of any of the three functions). Suppose further that $\lim_{x\to a} f(x)$ and $\lim_{x\to a} h(x)$ both exist and both equal L. Prove that $\lim_{x\to a} g(x)$ exists and equals L.

   (This is an example of a *squeeze theorem*: the function g is being squeezed between f and h near a.)

6. Prove that $\lim_{x\to 1} 1/(x-1)$ does not exist.

7. (a) Prove that if $\lim_{x\to a} g(x) = 0$, then $\lim_{x\to a} g(x)\sin(1/x) = 0$.
   (b) Suppose that $\lim_{x\to 0} g(x) = 0$ and $|h(x)| \le M$ for all x, for some $M \ge 0$. Prove that $\lim_{x\to 0} g(x)h(x) = 0$.

8. Here's the definition of $\lim_{x\to a} f(x) = L$, in symbols:

$$(\forall\varepsilon > 0)(\exists\delta > 0)(\forall x)((0 < |x-a| < \delta) \Rightarrow (|f(x) - L| < \varepsilon)). \qquad (\star)$$

   (a) Here's a very similar-looking statement (with some $<$'s changed to $\le$'s):

$$(\forall\varepsilon > 0)(\exists\delta > 0)(\forall x)((0 < |x-a| \le \delta) \Rightarrow (|f(x) - L| \le \varepsilon)). \qquad (\star\star)$$

      i. Does $(\star\star)$ imply $(\star)$?
      ii. Does $(\star)$ imply $(\star\star)$?

   (b) Here's another very similar-looking statement (with the order of quantifiers changed at the beginning):

$$(\exists\delta > 0)(\forall\varepsilon > 0)(\forall x)((0 < |x-a| < \delta) \Rightarrow (|f(x) - L| < \varepsilon)). \qquad (\star\star\star)$$

      i. Does $(\star\star\star)$ imply $(\star)$?
      ii. Does $(\star)$ imply $(\star\star\star)$?
      iii. If f satisfies $(\star\star\star)$, what must it look like near a?

An extra credit problem

Suppose that for each natural number n, A_n is a finite set of numbers, all in $(0,1)$, and that whenever $m \ne n$, A_n and A_m have no elements in common. Define

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is not in } A_n \text{ for any } n, \\ 1/n & \text{if } x \in A_n. \end{cases}$$

1. Prove that for all $a \in (0,1)$, $\lim_{x\to a} f(x) = 0$.

2. Show that this conclusion no longer necessarily holds if even one of the A_n is allowed to be infinite.

3. Why did I demand that A_n and A_m have no members in common for $n \ne m$?