Math 10850, Honors Calculus 1

Homework 6

Solutions

1. In each of the following cases, determine the limit L for the given a, and prove that it is indeed the limit by finding, for each $\varepsilon > 0$, a δ (probably depending on ε) such that $|f(x) - L| < \varepsilon$ for all x satisfying $0 < |x - a| < \delta$.

   (a) $f(x) = 100/x$, $a = 1$.

   Solution: We claim that the limit is 100.

   To prove this, suppose that $\varepsilon > 0$ is given. We want to find δ such that whenever $0 < |x - 1| < \delta$, we have $|100/x - 100| < \varepsilon$.

   Now $|100/x - 100| < \varepsilon$ is equivalent (after a little algebra) to $|x - 1|/|x| \leq \varepsilon/100$.

   Choose $\delta \leq 1/2$. Then $0 < |x - 1| < \delta$ implies $x \in (1/2, 3/2)$, so $|x| > 1/2$ and $100/|x| < 50$.

   Choose also $\delta \leq \varepsilon/50$. Then $0 < |x - 1| < \delta$ implies $|x - 1| < \varepsilon/50$.

   To get both conditions to hold, we choose $\delta = \min\{1/2, \varepsilon/50\}$; note that $\delta > 0$.

   For this δ, or any smaller positive δ, we have that if $0 < |x-1| < \delta$ then $|100/x-100| < 50(\varepsilon/50) = \varepsilon$.

   This proves that $\lim_{x\to 1} 100/x = 100$.

   (b) $f(x) = x^4 + 1/x$, arbitrary $a > 0$.

   Solution: We claim that the limit is $a^4 + 1/a$.

   To prove this, suppose that $\varepsilon > 0$ is given. We want to find δ such that whenever $0 < |x - a| < \delta$, we have $|(x^4 + 1/x - (a^4 + 1/a)| < \varepsilon$.

   Now (using triangle inequality frequently, and using that $a > 0$, $x^2 \geq 0$)

$$\begin{aligned}
|(x^4 + 1/x - (a^4 + 1/a)| &= |x^4 - a^4 + (1/x - 1/a)| \\
&\leq |x^4 - a^4| + |1/x - 1/a| \\
&= |(x-a)(x+a)(x^2+a^2)| + \left|\frac{a-x}{xa}\right| \\
&= |x-a||x+a||x^2+a^2| + \frac{|x-a|}{|x|a} \\
&= |x-a|\left(|x+a||x^2+a^2| + \frac{1}{|x|a}\right) \\
&\leq |x-a|\left((|x|+a)(x^2+a^2) + \frac{1}{|x|a}\right).
\end{aligned}$$

If $\delta \leq a/2$, that $0 < |x-a| < \delta$ implies $|x-a| < a/2$, which in turn implies $x \in (a/2, 3a/2)$, so $a/2 < |x| < 3a/2$. From this it follows that

$$(|x|+a)(x^2+a^2) + \frac{1}{|x|a} < \left(\frac{3a}{2}+a\right)\left(\frac{9a^2}{4}+a^2\right) + \frac{2}{a^2} = \frac{65a^3}{8} + \frac{2}{a^2}.$$

If also $\delta \leq \frac{\varepsilon}{\frac{65a^3}{8}+\frac{2}{a^2}}$ then, from the previous algebra, $0 < |x-a| < \delta$ implies $|(x^4+1/x-(a^4+1/a)| < \varepsilon$.
So if we take

$$\delta = \min\left\{1/2, \frac{\varepsilon}{\frac{65a^3}{8}+\frac{2}{a^2}}\right\}$$

then $0 < |x-a| < \delta$ implies $|(x^4+1/x-(a^4+1/a)| < \varepsilon$.
This proves that $\lim_{x\to a}(x^4+1/x) = a^4+1/a$.

2. Calculate the following limits, *not* directly from the definition, but instead using the various theorems we have proven about limits.

   (a) $\lim_{x\to 2}\frac{x^3-8}{x-2}$.

   Solution: The numerator factors as $(x-2)(x^2+2x+4)$. Since 2 is not in the domain of the function, it is legitimate to cancel the factors of $x-2$ above and below. This leads to

   $$\lim_{x\to 2}\frac{x^3-8}{x-2} = \lim_{x\to 2}(x^2+2x+4) = 12,$$

   the latter equality since x^2+2x+4 is rational, with 2 in its domain, so the limit is the value at 2.

   (b) $\lim_{x\to y}\frac{x^n-y^n}{x-y}$.

   Solution: Viewed as a function of x, with y a constant, the domain of this function is $\{x : x \neq y\}$. This means that we can divide through by $x-y$ without changing the limit (we are essentially multiplying the function by 1, with 1 written as $(1/(x-y))/(1/(x-y))$, which is valid as long as $x \neq y$). This leads to

   $$\lim_{x\to y}\frac{x^n-y^n}{x-y} = \lim_{x\to y}(x^{n-1}+yx^{n-2}+\cdots+y^{n-2}x+y^{n-1}).$$

   This latter is a rational function (in variable x) with y in the domain, so the limit is the value of the function at y, that is,

   $$y^{n-1}+yy^{n-2}+\cdots+y^{n-2}y+y^{n-1} \quad \text{or} \quad ny^{n-1}.$$

   (c) $\lim_{h\to 0}\frac{\sqrt{a+h}-\sqrt{a}}{h}$.

   Solution: Here the answer depends on a. If $a \leq 0$ then the function f defined by $f(h) = (\sqrt{a+h}-\sqrt{a})/h$ is not defined near 0 (because for any negative value of h, $a-h < 0$), so the limit does not exists.

If $a > 0$ then the function is defined near 0 (though not *at* 0), so we can study the limit. As long as $h \neq 0$ we have

$$\begin{aligned}\frac{\sqrt{a+h}-\sqrt{a}}{h} &= \left(\frac{\sqrt{a+h}-\sqrt{a}}{h}\right)\left(\frac{\sqrt{a+h}+\sqrt{a}}{\sqrt{a+h}+\sqrt{a}}\right) \\ &= \frac{(a+h)-a}{h(\sqrt{a+h}+\sqrt{a})} \\ &= \frac{h}{h(\sqrt{a+h}+\sqrt{a})} \\ &= \frac{1}{\sqrt{a+h}+\sqrt{a}}.\end{aligned}$$

So

$$\lim_{h\to 0}\frac{\sqrt{a+h}-\sqrt{a}}{h} = \lim_{h\to 0}\frac{1}{\sqrt{a+h}+\sqrt{a}} = \frac{1}{2\sqrt{a}}.$$

The last equality is obtained by direct evaluation, valid by the sum-product-reciprocal theorem, the composition theorem, and the fact (not yet proven) that the square root function is continuous on its domain.

3. For this question, the usual rules apply: if it is your understanding that a certain phenomenon holds in general, then you should provide a proof/justification that that is the case; if it does not hold in general, a single explicit counterexample is enough.

(a) If $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ both *do not exist*, can $\lim_{x\to a}(f(x)+g(x))$ exist?

Solution: Yes. Consider, for example,

$$f(x) = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \end{cases}$$

and

$$g(x) = \begin{cases} -1 & \text{if } x > 0 \\ 1 & \text{if } x < 0. \end{cases}$$

Certainly $\lim_{x\to 0} f(x)$ and $\lim_{x\to 0} g(x)$ do not exist. But $(f+g)(x) = 0$ unless $x = 0$ (at which point the sum is undefined), so $\lim_{x\to 0}(f+g)(x) = 0$.

(b) If $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ both *do not exist*, can $\lim_{x\to a} f(x)g(x)$ exist?

Solution: Yes. Consider, for example, exactly the same functions f and g from the previous part. $(fg)(x) = -1$ unless $x = 0$ (at which point the product is undefined), so $\lim_{x\to 0}(fg)(x) = -1$.

(c) If $\lim_{x\to a} f(x)$ and $\lim_{x\to a}(f(x)+g(x))$ both exist, must $\lim_{x\to a} g(x)$ exist?

Solution: Yes. If $\lim_{x\to a} f(x)$ exists, and $\lim_{x\to a}(f(x)+g(x))$ exists, then by the sum-product-reciprocal theorem,

$$\lim_{x\to a}((f(x)+g(x))-f(x)) = \lim_{x\to a} g(x)$$

exists.

(d) If $\lim_{x\to a} f(x)$ exists and $\lim_{x\to a} g(x)$ does not exist, can $\lim_{x\to a}(f(x)+g(x))$ exist?

Solution: No. If $\lim_{x\to a}(f(x)+g(x))$ existed then (by part b) $\lim_{x\to a} g(x)$ would also exist, a contradiction.

(e) If $\lim_{x\to a} f(x)$ exists and $\lim_{x\to a} f(x)g(x)$ exists, does it follow that $\lim_{x\to a} g(x)$ exists?

Solution: It is tempting to say "yes". If $\lim_{x\to a} f(x)$ exists, and $\lim_{x\to a} f(x)g(x)$ exists, then by the sum-product-reciprocal theorem,

$$\lim_{x\to a}(f(x)g(x))/f(x)) = \lim_{x\to a} g(x)$$

should exist; but this assumes that $\lim_{x\to a} f(x)$ is not zero. So to find a counter-example, we need to find functions f and g, and an a, with $\lim_{x\to a} f(x) = 0$, $\lim_{x\to a} f(x)g(x)$ existing, and $\lim_{x\to a} g(x)$ not existing.

Taking f to be the constant 0 function, g to be the function $g(x) = \sin(1/x)$ and $a = 0$ works nicely.

4. (a) Prove that $\lim_{x\to 0} f(x) = \lim_{x\to 0} f(x^3)$. **Clarification**: Show that if $\lim_{x\to 0} f(x) = L$ then $\lim_{x\to 0} f(x^3)$ exists and equals L.

Solution: There is an implicit assumption here, that both limits exists. We will show that if $\lim_{x\to 0} f(x) = L$ then $\lim_{x\to 0} f(x^3) = L$.

Suppose that $\lim_{x\to 0} f(x) = L$, so that for all $\varepsilon > 0$ there is $\delta > 0$ such that $0 < |x| < \delta$ implies $|f(x) - L| < \varepsilon$.

Now, given $\varepsilon > 0$, consider $0 < |x| < \delta^{1/3}$, where δ is exactly as in the last paragraph (we use here the as-yet-unproven fact that for every positive number t, there is a positive number s such that $s^3 = t$; we call this the cubed root of t, or $t^{1/3}$). Now $0 < |x| < \delta^{1/3}$ is the same as $-\delta^{1/3} < x < \delta^{1/3}$, $x \neq 0$, which is the same as $-\delta^3 < x^3 < \delta$, $x \neq 0$, which is the same as $0 < |x^3| < \delta$. In this range we have $|f(x^3) - L| < \varepsilon$, so that $\lim_{x\to 0} f(x^3) = L$, as claimed.

We could easily reverse this argument to show that if $\lim_{x\to 0} f(x^3) = L$ then $\lim_{x\to 0} f(x) = L$, and so prove that if either one of the two limits exist then they both do, and they are equal.

(b) Give an example where $\lim_{x\to 0} f(x^2)$ exists, but $\lim_{x\to 0} f(x)$ doesn't.

Solution: Let

$$f(x) = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0, \end{cases}$$

so that $f(x^2) = 1$ if $x \neq 0$ (and is undefined at $x = 0$). We have $\lim_{x\to 0} f(x^2) = 1$ but $\lim_{x\to 0} f(x)$ does not exist.

5. Let f, g, h be three functions, and let a be some real number. Suppose that there is some number $\Delta > 0$ such that on the interval $(a - \Delta, a + \Delta)$ it holds that $f(x) \leq g(x) \leq h(x)$ (except possibly at a, which might or might not be in the domains of any of the three functions). Suppose further that $\lim_{x\to a} f(x)$ and $\lim_{x\to a} h(x)$ both exist and both equal L. Prove that $\lim_{x\to a} g(x)$ exists and equals L.

(This is an example of a *squeeze theorem*: the function g is being squeezed between f and h near a.)

Solution: Let L be the common value of $\lim_{x\to a} f(x)$ and $\lim_{x\to a} h(x)$. We aim to show $\lim_{x\to a} g(x) = L$.

To that end, let $\varepsilon > 0$ be given. There is a $\delta_1 > 0$ such that for x satisfying $0 < |x-a| < \delta_1$, we have $|f(x) - L| < \varepsilon$, and there is a $\delta_2 > 0$ such that for x satisfying $0 < |x-a| < \delta_2$, we have $|h(x) - L| < \varepsilon$. Let $\delta > 0$ be any number no bigger than δ_1, δ_2 and Δ (e.g.,

$$\delta = \min\{\delta_1, \delta_2, \Delta\}.)$$

For x satisfying $0 < |x-a| < \delta$, we have both $|f(x) - L| < \varepsilon$ and $|h(x) - L| < \varepsilon$, in other words,

$$L - \varepsilon < f(x) \leq h(x) < L + \varepsilon.$$

But now, we know $f(x) \leq g(x) \leq h(x)$ for all such x (this is where we use $\delta \leq \Delta$); so in particular, for x satisfying $0 < |x-a| < \delta$ we have

$$L - \varepsilon < g(x) < L + \varepsilon$$

so $|g(x) - L| < \varepsilon$. This shows that $\lim_{x\to a} g(x) = L$.

6. Prove that $\lim_{x\to 1} 1/(x-1)$ does not exist.

Solution: Let L be given. We will show that $\lim_{x\to 1} 1/(x-1) \neq L$.

The main point is this: by taking x close enough to 1 (and, for definiteness, positive) we can make $1/(x-1)$ as large as we want, and in particular larger than $|L|+1$. Note specifically that if

$$x = \frac{|L|+2}{|L|+1}$$

then

$$\frac{1}{x-1} = |L|+1,$$

and that if $1 < y < x$ then $f(y) > f(x)$.

So, take $\varepsilon = 1/2$. Let $\delta > 0$ be given.

- If $\delta > 1/(|L|+1)$ then take $x = (|L|+2)/(|L|+1)$ (note that $0 < |x-1| < \delta$) to get $f(x) = |L|+1$, so $|f(x) - L| \geq 1/2$ (if $L \geq 0$, $|f(x) - L| = 1$, and if $L < 0$, $|f(x) - L| = 2|L| + 1 > 1$).
- If $\delta \leq 1/(|L|+1)$ then take $x = 1 + \delta/2$ (note that $0 < |x-1| < \delta$). Since $1 < x < (|L|+2)/(|L|+1)$, get $f(x) > f((|L|+2)/(|L|+1)) = |L|+1$, so again $|f(x) - L| \geq 1/2$.

This shows that $\lim_{x\to 1} 1/(x-1) \neq L$.

7. (a) Prove that if $\lim_{x\to a} g(x) = 0$, then $\lim_{x\to a} g(x)\sin(1/x) = 0$.

   Solution: Part (a) is implied by part (b), because $|\sin(1/x)| \leq 1$ for all $x \neq 0$, so we just prove part (b).

(b) Suppose that $\lim_{x\to 0} g(x) = 0$ and $|h(x)| \leq M$ for all x, for some $M \geq 0$. Prove that $\lim_{x\to 0} g(x)h(x) = 0$.

Solution: Suppose that $\lim_{x\to 0} g(x) = 0$ and $|h(x)| \leq M$ for all x, for some $M \geq 0$. We claim that $\lim_{x\to 0} g(x)h(x) = 0$.
Let $\varepsilon > 0$ be given. We need to find $\delta > 0$ such that $0 < |x| < \delta$ implies $|g(x)h(x)| < \varepsilon$. But

$$|g(x)h(x)| = |g(x)||h(x)| \leq M|g(x)|,$$

so it is enough to find a $\delta > 0$ such that $0 < |x| < \delta$ implies $M|g(x)| < \varepsilon$, or equivalently $|g(x)| < \varepsilon/M$. Now because $\lim_{x\to 0} g(x) = 0$ (and because $\varepsilon/M > 0$), there is such a δ.

8. Here's the definition of $\lim_{x\to a} f(x) = L$, in symbols:

$$(\forall \varepsilon > 0)(\exists \delta > 0)(\forall x)((0 < |x - a| < \delta) \Rightarrow (|f(x) - L| < \varepsilon)). \qquad (\star)$$

(a) Here's a very similar-looking statement (with some $<$'s changed to $\leq$'s):

$$(\forall \varepsilon > 0)(\exists \delta > 0)(\forall x)((0 < |x - a| \leq \delta) \Rightarrow (|f(x) - L| \leq \varepsilon)). \qquad (\star\star)$$

i. Does $(\star\star)$ imply $(\star)$?

Solution: Yes. Suppose we know $(\star\star)$. We aim to prove $(\star)$. Let $\varepsilon > 0$ be given. Apply $(\star\star)$ with "ε" replaced by "$\varepsilon/2$" (valid, since $\varepsilon/2 > 0$). We get that there is $\delta > 0$ such that for all x,

$$[0 < |x - a| \leq \delta] \Rightarrow [|f(x) - L| \leq \varepsilon/2].$$

But then it is certainly true that

$$[0 < |x - a| < \delta] \Rightarrow [|f(x) - L| \leq \varepsilon/2],$$

since all x satisfying $0 < |x - a| \leq \delta$ also satisfy $0 < |x - a| < \delta$. But then, further, it is certainly true that

$$[0 < |x - a| < \delta] \Rightarrow [|f(x) - L| < \varepsilon],$$

since $\varepsilon/2 < \varepsilon$. So $(\star)$ holds, since for all $\varepsilon > 0$ we have found a $\delta > 0$ such that for all x, $[0 < |x - a| < \delta] \Rightarrow [|f(x) - L| < \varepsilon]$.

ii. Does $(\star)$ imply $(\star\star)$?

Solution: Yes. Suppose we know $(\star)$. We aim to prove $(\star\star)$. Let $\varepsilon > 0$ be given. Apply $(\star)$ to find a $\delta' > 0$ such that for all x, $[0 < |x - a| < \delta'] \Rightarrow [|f(x) - L| < \varepsilon]$. Take $\delta = \delta'/2$. If $0 < |x - a| \leq \delta$, then it is certainly true that $0 < |x - a| < \delta'$, so it follows that $[|f(x) - L| < \varepsilon]$, which in turn implies $[|f(x) - L| \leq \varepsilon]$. Hence $(\star\star)$ is true.

NOTE: This exercise shows that there is no change to the definition of a limit, if we replace "$< \delta$" and/or "$< \varepsilon$" with "$\leq \delta$" and/or "$\leq \varepsilon$"

(b) Here's another very similar-looking statement (with the order of quantifiers changed at the beginning):

$$(\exists\delta > 0)(\forall\varepsilon > 0)(\forall x)((0 < |x - a| < \delta) \Rightarrow (|f(x) - L| < \varepsilon)). \qquad (\star\star\star)$$

i. Does $(\star\star\star)$ imply $(\star)$?

Solution: Yes. Suppose we know $(\star\star\star)$. Let $\varepsilon > 0$ be given. By $(\star\star\star)$ we know that there is a particular $\delta > 0$ (which has nothing to do with ε), such that for any particular $\varepsilon' > 0$, whenever we have $0 < |x - a| < \delta$ we also have $|f(x) - L| < \varepsilon'$. In particular that means that for our specified $\varepsilon > 0$, whenever we have $0 < |x - a| < \delta$ we also have $|f(x) - L| < \varepsilon$. So $(\star)$ holds.

ii. Does $(\star)$ imply $(\star\star\star)$?

Solution: No. To show this, all we need is a single counter-example. Consider the function $f(x) = x$, and take $a = 0$, $L = 0$. $(\star)$ certainly holds:

$$(\forall\varepsilon > 0)(\exists\delta > 0)(\forall x)([0 < |x| < \delta] \Rightarrow [|x| < \varepsilon]);$$

indeed, we may take $\delta = \varepsilon$.
However, $(\star\star\star)$ claims

$$(\exists\delta > 0)(\forall\varepsilon > 0)(\forall x)([0 < |x| < \delta] \Rightarrow [|x| < \varepsilon]).$$

We claim this is false. Indeed, given any $\delta > 0$, take $\varepsilon = \delta/2$. The statement

$$(\forall x)([0 < |x| < \delta] \Rightarrow [|x| < \delta/2])$$

is clearly false, as witnessed for example by $x = 3\delta/4$.

iii. If f satisfies $(\star\star\star)$, what must it look like near a?
Solution: If $(\star\star\star)$ holds, then there is some number $\delta > 0$ such that for any x in both $(a - \delta, a)$ and $(a, a + \delta)$, it holds that for *any* $\varepsilon > 0$, $|f(x) - L| < \varepsilon$. This says that $f(x) = L$ on both these intervals. (**Proof**: Indeed, suppose there is some $x_0 \in (a - \delta, a) \cup (a, a + \delta)$ with $f(x_0) \neq L$. Then $|f(x_0) - L| > 0$. Picking any $\varepsilon > 0$ that is smaller than $|f(x_0) - L|$, we *cannot* have $|f(x_0) - L| < \varepsilon$.)
So: if f satisfies $(\star\star\star)$, near a it must be constant, and take the value L.