Math 10850, Honors Calculus 1

Homework 7

Due in class Friday November 1

General and specific notes on the homework

All the notes from homework 1 still apply!

Just like the last homework assignment, this one is based on definitions — the definition of a limit, and the definition of continuity. But for the most part, I'm not asking you tackle the questions using the ε-δ formalism. Instead, I want you to give complete and coherent arguments, based on theorems we have seen in class, and based on what by now should be your much clearer understanding of the notion of limit.

As an example of what I'm talking about: I'm quite happy for you to take state, *without a formal ε-δ proof*, that the function $f : [0,2] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 0 & \text{if } x \leq 1 \\ x^2 & \text{if } x > 1. \end{cases}$$

if continuous at all points other than 1, and that it is left continuous at 1.

Mathematics is a subject in which ideas build up upon each other. We could get very little done mathematically if we insisted on always going back to the axioms/back to the definitions. Instead, we should take the things we prove from the axioms/definitions as truths that we can subsequently use, without having to reprove them. If you draw on things that you already know, you will find that the homework goes much faster.

In particular, all questions from Question 8 on are about the Intermediate Value Theorem, and you will find life very difficult indeed unless you are willing to *use* the IVT as a tool in answering these questions!

There are only a few places where I am insisting on ε-δ proofs. I've clearly indicated those places.

In an attempt to shorten the homework, I've made a few problems optional. Not doing those questions will not harm your preparation for quizzes and/or exams. I strongly encourage you to work on these problems, as time permits, but they won't get graded.

Reading for this homework

This homework covers one sided limits (Section 6.5 of the course notes), and continuity through the intermediate value theorem (Sections 7.1 through 7.3 of the course notes). In Spivak this is part of Chapter 5, all of Chapter 6 and part of Chapter 7.

Assignment

1. For each of these functions f, EITHER find a function F, which is continuous at *all* real numbers, and for which $f(x) = F(x)$ for all x in the domain of f, OR show that no such function F exists. **Reiterating the earlier instruction**: You don't need to get bogged down in ε-δ formalism here; give a clear example of an F if one such exists, and a clear explanation of why no such F exists otherwise.

   (a) $f(x) = \frac{x^2-4}{x-2}$.

   (b) $f(x) = \frac{|x|}{x}$.

   (c) $f(x) = 0$, Domain$(f) = \{$irrational numbers$\}$.

2. For this question, I'm expecting a detailed ε-δ argument. Note that part (b) implies part (a); you might choose to do part (a) as a warm-up, or jump straight to part (b) and ignore part (a).

   (a) Suppose that f is a function satisfying $|f(x)| \leq |x|$ for all x. Show that f is continuous at 0.

   (b) Suppose that g is continuous at 0, that $g(0) = 0$, and that $|f(x)| \leq |g(x)|$ for all x. Prove that f is continuous at 0.

3. **OPTIONAL!** Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ that is continuous at 0 but not continuous at any other point. (**Note**: by shifting, you could then easily find, for any fixed real a, a function that is continuous at a but not continuous at any other point.)

4. Give an example of a function f such that f is continuous nowhere, but $|f|$ is continuous everywhere. (Given examples we have seen in class, this should be *very* easy. To re-iterate the introductory note, I'm not looking for ε-δ formalism here, but rather a

   - concise,
   - complete,
   - coherent,
   - convincing &
   - correct

   explanation; and the same goes for the remaining questions).

5. Find a function f which is continuous at all points on the real line except $1, 1/2, 1/3, \ldots$, and 0, and has the property that none of $\lim_{x\to 1} f(x)$, $\lim_{x\to 1/2} f(x)$, $\lim_{x\to 1/3} f(x)$, etc., exist, nor $\lim_{x\to 0} f(x)$.

6. (a) Prove that if f is continuous at ℓ, and if $\lim_{x\to a} g(x) = \ell$, then $\lim_{x\to a} f(g(x)) = f(\ell)$ (**SO**: "a continuous function can be passed inside a limit"). (**Hint**: For this you could go right back to the definitions, or you could introduce the function G defined by

$$G(x) = \begin{cases} g(x) & \text{if } x \neq a \\ \ell & \text{if } x = a.) \end{cases}$$

(b) **OPTIONAL!** Show that if we *do not* assume continuity of f at ℓ, then it is not generally true that

$$\lim_{x\to a} f(g(x)) = f(\lim_{x\to a} g(x)).$$

Hint: Consider the function f defined by

$$f(x) = \begin{cases} 0 & \text{if } x \neq \ell \\ 1 & \text{if } x = \ell. \end{cases}$$

7. **OPTIONAL!** (but easy, hopefully. The first part came up in class on Friday before break.)

   (a) Prove that if f is continuous on $[a, b]$ then there is a function $g : \mathbb{R} \to \mathbb{R}$ that is continuous on all of $\mathbb{R}$, and which satisfies $g(x) = f(x)$ for all $x \in [a, b]$. (I.e., every continuous function on a closed interval can be extended to a continuous function on the reals). **Hint**: Don't be too fancy with the definition of g to the left of a or to the right of b — the obvious idea works. And again, you don't need to use an ε-δ argument; just know that you *could*, if needed.

   (b) Show, by an example, that if f is only continuous on the *open* interval (a, b), then such a g need not necessarily exist. **NB**: you just need to give a coherent explanation of why your example works; an ε-δ proof is not required.

8. For each of the following polynomials, find (with justification!) an integer n such that $p(x) = 0$ for some $x \in (n, n + 1)$, or prove that no such n exists.

   (a) $p(x) = x^5 + 5x^4 + 2x + 1$

   (b) $q(x) = x^5 + x + 100$

   (c) $r(x) = x^4 + 4x^3 + 7x^2 + 6x + 3$

9. Suppose that f is a continuous function on $[a, b]$ (some $a < b$), and that f only takes on rational values. What can you conclude about f? Justify!

10. For this question, $f : [0, 1] \to [0, 1]$ is a continuous function on $[0, 1]$ that only takes on values between 0 and 1. Pictures will help for each part.

    (a) Prove that there is a number x, $0 \leq x \leq 1$, such that $f(x) = x$.

    (b) The previous part shows that f crosses the diagonal $(0, 0)$ to $(1, 1)$ of the unit square. Show that it also crosses the other diagonal, the one from $(0, 1)$ to $(1, 0)$. That is, show that there is an x, $0 \leq x \leq 1$, such that $(x, f(x))$ lies on the line $x + y = 1$.

    (c) More generally, prove that if g is continuous on $[0, 1]$ with EITHER $g(0) = 0$ and $g(1) = 1$ OR $g(0) = 1$ and $g(1) = 0$ then there is a number x, $0 \leq x \leq 1$, such that $f(x) = g(x)$.

11. **OPTIONAL!** Let f be a continuous function on $[a, b]$ with $f(a) < 0 < f(b)$. We proved the intermediate value theorem in class by showing that there is a smallest x in $[a, b]$ with $f(x) = 0$. If there is more than one x in $[a, b]$ with $f(x) = 0$, is there necessarily a second smallest?

Extra credit problems

1. (Easy, no credit for this, don't tell me the answer) Is there a continuous function that takes on every real value exactly once (i.e., is there $f : \mathbb{R} \to \mathbb{R}$ such that for each $y \in \mathbb{R}$, there is *exactly one* $x \in \mathbb{R}$ such that $f(x) = y$)?

2. Is there a continuous function that takes on every real value exactly twice? Justify, *clearly and coherently*!

3. Is there a continuous function that takes on every real value exactly three times? Justify, *clearly and coherently*!