Math 10850, Honors Calculus 1

Homework 7

Solutions

1. For each of these functions f, EITHER find a function F, which is continuous at *all* real numbers, and for which $f(x) = F(x)$ for all x in the domain of f, OR show that no such function F exists. **Reiterating the earlier instruction**: You don't need to get bogged down in ε-δ formaIism here; give a clear example of an F if one such exists, and a clear explanation of why no such F exists otherwise.

   (a) $f(x) = \frac{x^2-4}{x-2}$.

   Solution: Away from $x = 2$ (the only real not in the domain of f), f is equal to $x + 2$. This function is continuous at all real numbers; so $F(x) = x + 2$ works as an answer here. Another way of writing it:

   $$F(x) = \begin{cases} \frac{x^2-4}{x-2} & \text{if } x \neq 2, \\ 4 & \text{if } x = 2. \end{cases}$$

   (b) $f(x) = \frac{|x|}{x}$.

   Solution: No F exists. The domain of f is all reals except 0, and $\lim_{x\to 0} f(x)$ does not exist. (Proof [[[NOT REQUIRED!!!]]]: given L, take $\varepsilon = 1/4$. Given $\delta > 0$, in the range $0 < |x| < \delta$ there are some x for which $f(x) = 1$ ($x = \delta/2$, say), and some x for which $f(x) = -1$ ($x = \delta/2$, say). Sine 1 and -1 are distance two apart, it is not possible for *both* of these values to be in the interval $(L - 1/4, L + 1/4)$ (which has length only 1/2). So for all L, there is an $\varepsilon > 0$ (in particular, $\varepsilon = 1/4$) such that for all $\delta > 0$ there is an x (either $x = \delta/2$ or $x = -\delta/2$) such that $0 < |x| < \delta$ and $|f(x) - L| \geq \varepsilon$. This is exactly what it means for the limit not to exist.). Hence no matter what we choose for $F(0)$, the function F that extends f to all reals will not be continuous at 0.

   (c) $f(x) = 0$, Domain(f) = {irrational numbers}.

   Solution: The constant function F with $F(x) = 0$ for all x works here, fairly trivially.

2. For this question, I'm expecting a detailed ε-δ argument. Note that part (b) implies part (a); you might choose to do part (a) as a warm-up, or jump straight to part (b) and ignore part (a).

(a) Suppose that f is a function satisfying $|f(x)| \leq |x|$ for all x. Show that f is continuous at 0.

Solution: Suppose $|f(x)| \leq |x|$ for all x. We claim that $\lim_{x\to 0} f(x) = 0$. Indeed, given $\varepsilon > 0$, take $\delta = \varepsilon$. If $0 < |x| < \delta$ then $|f(x)| \leq |x| < \delta = \varepsilon$. This proves that $\lim_{x\to 0} f(x) = 0$. To conclude that f is cts at 0, note that, as pointed out in the question, applying "$|f(x)| \leq |x|$ for all x" at $x = 0$ gives $|f(0)| \leq 0$ so $f(0) = 0$, and hence $\lim_{x\to 0} f(x) = 0$ implies $\lim_{x\to 0} f(x) = f(0)$, i.e., f is cts at 0.

(b) Suppose that g is continuous at 0, that $g(0) = 0$, and that $|f(x)| \leq |g(x)|$ for all x. Prove that f is continuous at 0.

Solution: The conditions $|f(x)| \leq |g(x)|$ for all x and $g(0) = 0$ together imply $f(0) = 0$, so we just need to show $\lim_{x\to 0} f(x) = 0$.

Let $\varepsilon > 0$ be given. Because g is continuous at 0 (and $g(0) = 0$) there is a $\delta > 0$ such that $0 < |x| < \delta$ implies $|g(x) - g(0)| = |g(x)| < \varepsilon$. But then using $|f(x)| \leq |g(x)|$ for all x, we see that $0 < |x| < \delta$ implies $|f(x)| \leq |g(x)| < \varepsilon$. This shows $\lim_{x\to 0} f(x) = 0$.

3. **OPTIONAL!** Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ that is continuous at 0 but not continuous at any other point. (**Note**: by shifting, you could then easily find, for any fixed real a, a function that is continuous at a but not continuous at any other point.)

Solution: Here's one possible solution; there are many others.

Consider the function

$$f(x) = \begin{cases} x & f \text{ is rational} \\ 0 & f \text{ is irrational.} \end{cases}$$

Certainly $|f(x)| \leq |x|$ for all x, and so, by the result of the previous part, f is continuous at 0.

Using the same proof from class as we used for Dirichlet's function (1 on rationals, 0 on irrationals), we get that f is not continuous at any non-zero point. Specifically: let a be non-zero, say initially $a > 0$. Let L be given. Take $\varepsilon = a/100$. Given $\delta > 0$, for x in the range $0 < |x - a| < \delta$ there are both irrationals (where f is 0) and rationals. If $a/2$ is in the range $0 < |x - a| < \delta$ then there is some rational in the range between $a/2$ and a, on which f takes value at least $a/2$. If If $a/2$ is *not* in the range $0 < |x - a| < \delta$ then $\delta \leq a/2$, and so certainly in the range $0 < |x - a| < \delta$ there is a rational on which f takes value at least $a/2$. So either way, there is an x_1 in the range $0 < |x - a| < \delta$ on which f takes value 0, and an x_2 in the same range on which f takes value at least $a/2$. Since these two values are at least $a/2$ apart, and the interval $(l - \varepsilon, L + \varepsilon)$, having length $2\varepsilon = a/50$ cannot contain both $f(x_1)$ and $f(x_2)$, the function f does not tend to a limit L near a; and since was arbitrary, there is no limit. A similar argument can be made if $a < 0$.

4. Give an example of a function f such that f is continuous nowhere, but $|f|$ is continuous everywhere. (Given examples we have seen in class, this should be *very* easy. To re-iterate the introductory note, I'm not looking for ε-δ formalism here, but rather a

- concise,
- complete,

- coherent,
- convincing &
- correct

explanation; and the same goes for the remaining questions).

Solution: Here's one possible solution; there are many others.

Consider the function

$$f(x) = \begin{cases} 1 & f \text{ is rational} \\ -1 & f \text{ is irrational.} \end{cases}$$

Using the same proof from class as we used for Dirichlet's function (1 on rationals, 0 on irrationals), we get that f is continuous at no point. But $|f|$ is a constant function (it is always 1), so is continuous everywhere.

5. Find a function f which is continuous at all points on the real line except $1, 1/2, 1/3, \ldots$, and 0, and has the property that none of $\lim_{x\to 1} f(x)$, $\lim_{x\to 1/2} f(x)$, $\lim_{x\to 1/3} f(x)$, etc., exist, nor $\lim_{x\to 0} f(x)$.

   Solution: There are many possible solutions; here is one.

   Take the function which is defined to be

   - 1 on the interval $[1, \infty)$
   - 2 on the interval $[1/2, 1)$
   - 3 on the interval $[1/3, 1/2)$
   - and in general takes constant value n on the interval $[1/n, 1/(n-1))$, $(n \in \mathbb{N})$,

   takes the value 0 on $(-\infty, 0)$, and is undefined at 0. As before, f is cts at any point in any open interval of the form $(-\infty, 0)$, $(1, \infty)$ or $(1/n, 1/(n-1))$, $n \in \mathbb{N}$, and discontinuous at 1, 1/2, 1/3, etc.. And it's discontinuous at 0, not being defined there. So f has discontinuities exactly at $1, 1/2, 1/3, \ldots$ and 0.

6. (a) Prove that if f is continuous at ℓ, and if $\lim_{x\to a} g(x) = \ell$, then $\lim_{x\to a} f(g(x)) = f(\ell)$ (**SO**: "a continuous function can be passed inside a limit"). (**Hint**: For this you could go right back to the definitions, or you could introduce the function G defined by

   $$G(x) = \begin{cases} g(x) & \text{if } x \neq a \\ \ell & \text{if } x = a.) \end{cases}$$

   Solution: Recall that in class we proved: if g is cts at a and f is cts at $g(a)$ then $f \circ g$ is cts at a.
   As the hint suggests, define

   $$G(x) = \begin{cases} g(x) & \text{if } x \neq a \\ \ell & \text{if } x = a. \end{cases}$$

We certainly have $\lim_{x\to a} G(x) = \lim_{x\to a} g(x)$ (because G agrees with g away from a), and since $\lim_{x\to a} g(x) = \ell$ and $G(a) = \ell$ we have $\lim_{x\to a} G(x) = G(a)$, that is, G is cts at a. We also have that f is cts at $G(a)$ (since $G(a) = \ell$). So $f \circ G$ is cts at a, i.e., $\lim_{x\to a} f(G(x)) = f(G(a))$. But $f(G(a)) = f(\ell)$, and $\lim_{x\to a} f(G(x)) = \lim_{x\to a} f(g(x))$ (since g agrees with G off a). So, as claimed,

$$\lim_{x\to a} f(g(x)) = f(\ell).$$

(b) **OPTIONAL!** Show that if we *do not* assume continuity of f at ℓ, then it is not generally true that

$$\lim_{x\to a} f(g(x)) = f(\lim_{x\to a} g(x)).$$

Hint: Consider the function f defined by

$$f(x) = \begin{cases} 0 & \text{if } x \neq \ell \\ 1 & \text{if } x = \ell. \end{cases}$$

Solution (part b): Consider, as suggested, the function

$$f(x) = \begin{cases} 0 & \text{if } x \neq \ell \\ 1 & \text{if } x = \ell. \end{cases}$$

Certainly this function is not cts at ℓ. In order to show that it not in general true that

$$\lim_{x\to a} f(g(x)) = f(\lim_{x\to a} g(x))$$

if we don't assume that f is cts at ℓ, we will use the f defined above. We need to find a function g such that

$$\lim_{x\to a} g(x) = \ell, \tag{1}$$

so that $f(\lim_{x\to a} g(x)) = f(\ell) = 1$, but that

$$\lim_{x\to a} f(g(x)) \neq 1. \tag{2}$$

We can achieve this by making g a cts function that takes the value ℓ at a (so that (1) holds), but does not take the value a at any other input (so that $f(g(x)) = 0$ for all $x \neq a$, and so $\lim_{x\to a} f(g(x)) = 0$, making (2) hold).
There are many such functions, for example $g(x) = x - a + \ell$.

7. **OPTIONAL!** (but easy, hopefully. The first part came up in class on Friday before break.)

(a) Prove that if f is continuous on $[a, b]$ then there is a function $g : \mathbb{R} \to \mathbb{R}$ that is continuous on all of $\mathbb{R}$, and which satisfies $g(x) = f(x)$ for all $x \in [a, b]$. (I.e., every continuous function on a closed interval can be extended to a continuous function on the reals). **Hint**: Don't be too fancy with the definition of g to the left of a or to the right of b — the obvious idea works. And again, you don't need to use an ε-δ argument; just know that you *could*, if needed.

Solution: We have defined "cts on $[a, b]$" to mean: for all $c \in (a, b)$, $\lim_{x\to c} f(x) = f(c)$, and also both of $\lim_{x\to a^+} f(x) = f(a)$ and $\lim_{x\to b^-} f(x) = f(b)$.
Define

$$g(x) = \begin{cases} f(a) & \text{if } x \leq a \\ f(x) & \text{if } a \leq x \leq b \\ f(b) & \text{if } b \leq x. \end{cases}$$

It is easy to check that g is continuous at every point in each of the intervals $(-\infty, a)$, (b, ∞) and (a, b). The latter is by hypothesis on f. For the two former, let's argue that for $c < a$, $\lim_{x\to c} g(x) = g(c)$. Given $\varepsilon > 0$, let $\delta > 0$ be any number small enough that $(c - \delta, c + \delta) \subseteq (-\infty, a)$ (such a δ exists because $c < a$; $\delta = (a - c)/2$ works). For all x in this interval, and so in particular for all x, $0 < |x - a| < \delta$, we have $g(x) = f(a) = g(c)$, so $|g(x) - g(c)| = 0 < \varepsilon$. This verifies $\lim_{x\to c} g(x) = g(c)$; continuity on (b, ∞) is identical.

It remains to check that g is cts at a and b. We just look at a; the case of b is identical. We know $\lim_{x\to a^+} f(x) = f(a)$ so $\lim_{x\to a^+} g(x) = g(a)$. Using a similar argument to the previous paragraph, it is very easy to check that $\lim_{x\to a^-} g(x) = g(a)$. So $\lim_{x\to a^+} g(x) = \lim_{x\to a^-} g(x) = g(a)$, and so $\lim_{x\to a} g(x) = g(a)$, and g is cts at a.

(b) Show, by an example, that if f is only continuous on the *open* interval (a, b), then such a g need not necessarily exist. **NB**: you just need to give a coherent explanation of why your example works; an ε-δ proof is not required.

Solution: There are many examples that work here. Here is one. Define $f(x) = 1/(x^2 - 1)$ on the interval $(-1, 1)$. This is continuous, but neither $\lim_{x\to -1^+} f(x)$ nor $\lim_{x\to 1^-} f(x)$ exist (easy exercise, I won't give the details). Now for f to be extended to a function g that is cts on the whole real line, it is necessary for both $\lim_{x\to -1} g(x)$ and $\lim_{x\to 1} g(x)$ to exist, which requires $\lim_{x\to -1^+} f(x)$ and $\lim_{x\to 1^-} f(x)$ to exist. So f can't be extended to a function that is cts on the whole real line.

8. For each of the following polynomials, find (with justification!) an integer n such that $p(x) = 0$ for some $x \in (n, n + 1)$, or prove that no such n exists.

(a) $p(x) = x^5 + 5x^4 + 2x + 1$

Solution: $p(-5) = -9$ and $p(-4) = 249$ (found by a little trial-and-error), so by the intermediate value theorem there is $x \in (-5, -4)$ with $p(x) = 0$ (note that p is continuous on $[-5, -4]$).

(b) $q(x) = x^5 + x + 100$

Solution: $q(-3) = -146$ and $q(-2) = 66$ (found by a little trial-and-error), so by the intermediate value theorem there is $x \in (-3, -2)$ with $p(x) = 0$ (note that q is continuous on $[-3, -2]$).

(c) $r(x) = x^4 + 4x^3 + 7x^2 + 6x + 3$

Solution: Here a little trial-and-error suggests that $r(x) > 0$ for all integers. One way to prove this is to observe:

$$r(x) = x^4 + 4x^3 + 7x^2 + 6x + 3 = (x + 1)^4 + x^2 + 2x + 2 = (x + 1)^4 + (x + 1)^2 + 1$$

and since $(x+1)^4 \geq 0$, $(x+1)^2 \geq 0$ for all $x \in \mathbb{R}$, we have $r(x) \geq 1$ for all $x \in \mathbb{R}$.

9. Suppose that f is a continuous function on $[a, b]$ (some $a < b$), and that f only takes on rational values. What can you conclude about f? Justify!

   Solution: We can conclude that f is constant. For suppose not; then there are two rational numbers r_1 and r_2 such that for some c, d, $a \leq c < d \leq b$, $f(c) = r_1$ and $f(d) = r_2$. By the full-strength intermediate value theorem, in the interval $[c, d]$ f takes on all values between r_1 and r_2, and this includes some irrational numbers, contradicting the fact that f only takes on rational values.

10. For this question, $f : [0, 1] \to [0, 1]$ is a continuous function on $[0, 1]$ that only takes on values between 0 and 1. Pictures will help for each part.

    (a) Prove that there is a number x, $0 \leq x \leq 1$, such that $f(x) = x$.

    Solution: If $f(0) = 0$ or if $f(1) = 1$ then we are done, so we may assume that $f(0) > 0$ and that $f(1) < 1$.
    Consider the function defined by $h(x) = f(x) - x$, which is continuous on $[0, 1]$. We have $h(0) = f(0) - 0 > 0$ and $h(1) = f(1) - 1 < 0$. So by the intermediate value theorem there is x, $0 < x < 1$ such that $h(x) = 0$. But for such an x we have $f(x) - x = 0$, or $f(x) = x$.

    (b) The previous part shows that f crosses the diagonal $(0, 0)$ to $(1, 1)$ of the unit square. Show that it also crosses the other diagonal, the one from $(0, 1)$ to $(1, 0)$. That is, show that there is an x, $0 \leq x \leq 1$, such that $(x, f(x))$ lies on the line $x + y = 1$.

    Solution: If $f(0) = 1$ or if $f(1) = 0$ then we are done, so we may assume that $f(0) < 1$ and that $f(1) > 0$.
    Consider the function defined by $h(x) = f(x) - (1 - x)$, which is continuous on $[0, 1]$. We have $h(0) = f(0) - 1 < 0$ and $h(1) = f(1) - 0 > 0$. So by the intermediate value theorem there is x, $0 < x < 1$ such that $h(x) = 0$. But for such an x we have $f(x) - (1 - x) = 0$, or $f(x) = 1 - x$, and so $(x, f(x))$ lies on the line $x + y = 1$.

    (c) More generally, prove that if g is continuous on $[0, 1]$ with EITHER $g(0) = 0$ and $g(1) = 1$ OR $g(0) = 1$ and $g(1) = 0$ then there is a number x, $0 \leq x \leq 1$, such that $f(x) = g(x)$.

    Solution: First consider the case $g(0) = 0$ and $g(1) = 1$. If either $f(0) = 0$ or $f(1) = 1$ then we are done, so we may assume that $f(0) > 0$ and that $f(1) < 1$.
    Consider the function defined by $h(x) = f(x) - g(x)$, which is continuous on $[0, 1]$. We have $h(0) = f(0) - g(0) > 0$ and $h(1) = f(1) - g(1) < 0$. So by the intermediate value theorem there is x, $0 < x < 1$ such that $h(x) = 0$. But for such an x we have $f(x) - g(x) = 0$, or $f(x) = g(x)$.
    Next consider the case $g(0) = 1$ and $g(1) = 0$. If either $f(0) = 1$ or $f(1) = 0$ then we are done, so we may assume that $f(0) < 1$ and that $f(1) > 0$.
    Consider the function defined by $h(x) = f(x) - g(x)$, which is continuous on $[0, 1]$. We have $h(0) = f(0) - g(0) < 0$ and $h(1) = f(1) - g(1) > 0$. So by the intermediate value theorem there is x, $0 < x < 1$ such that $h(x) = 0$. But for such an x we have $f(x) - g(x) = 0$, or $f(x) = g(x)$.

11. **OPTIONAL!** Let f be a continuous function on $[a, b]$ with $f(a) < 0 < f(b)$. We proved the intermediate value theorem in class by showing that there is a smallest x in $[a, b]$ with $f(x) = 0$. If there is more than one x in $[a, b]$ with $f(x) = 0$, is there necessarily a second smallest?

    Solution: Not necessarily. Consider $a = -1$, $b = 2/\pi$ and $f(x)$ defined piece-wise:

$$f(x) = \begin{cases} x & \text{if } x \leq 0 \\ x \sin(1/x) & \text{if } x > 0. \end{cases}$$

    This is continuous on $[-1, 2/\pi]$, and satisfies $f(a) < 0 < f(b)$. The smallest x in $[a, b]$ with $f(x) = 0$ is $x = 0$, and there are other inputs to the function that give output 0. But there is no second-smallest such input; for any $\delta > 0$ the interval $(0, \delta)$ contains infinitely many numbers at which the function evaluates to 0.