Math 10850, Honors Calculus 1

Homework 9

Due in class Friday November 15

General and specific notes on the homework

All the notes from homework 1 still apply!

Reading for this homework

This homework covers the derivative. You should read Section 8 of the class notes (but there are no questions here on the chain rule), and/or Spivak Chapter 9 and 10.

Assignment

1. When we motivated the definition of derivative via instantaneous velocity, we got to the expression $\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$, and when we motivated it by slope of tangent line, we ended up with $\lim_{b\to a}\frac{f(b)-f(a)}{b-a}$. It seems intuitively clear that these two expressions are the same. This question asks you to prove this, directly from the definition of limit. It also asks you to show that the expression we got at the end of the proof that differentiability implies continuity, namely $\lim_{h\to 0} f(a+h) = f(a)$, does indeed imply that f is continuous at a, even though the expression looks a little bit different from the definition of continuity.

   (a) Let g be a function defined near a. Suppose that $\lim_{b\to a} g(b)$ exists and equals L. Prove that $\lim_{h\to 0} g(a+h)$ exists and also equal L.[1]

   (b) **OPTIONAL!** (Almost exactly the same as part (a)) Let g be a function defined near a. Suppose that $\lim_{h\to 0} g(a+h)$ exists and equals L. Prove that $\lim_{b\to a} g(b)$ exists and also equal L.

   (c) Parts (a) and (b) together show that if g be a function defined near a, then if one of $\lim_{h\to 0} g(a+h)$, $\lim_{b\to a} g(b)$ exists, they are equal. Apply this to show that our two definitions of the derivative are equivalent. That is, show that if f is a function defined at and near a, and if either one of $\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$, $\lim_{b\to a}\frac{f(b)-f(a)}{b-a}$ exists, then they both exist and are equal. (This should be a simple matter of finding the right choice of function g).

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[1] You <u>must</u> use the ε-δ definition of the limit here. Start by supposing that an $\varepsilon > 0$ is given. You

- **Know**: that for any $\varepsilon' > 0$ there is $\delta' > 0$ such that $0 < |b-a| < \delta'$ implies $|g(b) - L| < \varepsilon'$.

You

- **Want**: that there is $\delta > 0$ such that $0 < |h| < \delta$ implies $|g(a+h) - L| < \varepsilon$.

Explain in your proof how to get where you **want** from what you **know**.

(d) **OPTIONAL!** (Very similar to part (c)). For f defined at and near a, we defined "f continuous at a" to mean "$\lim_{x\to a} f(x) = f(a)$". Show that an equivalent definition is "$\lim_{h\to 0} f(a+h) = f(a)$".

2. Let f be defined by $f(x) = \frac{x+1}{x-1}$. *Directly from the definition*[2] calculate $f'(a)$ for each $a \neq -1$.

3. (a) Prove, *directly from the definition* that if $f_3(x) = x^{1/3}$ then for all $a \neq 0$,

$$f_3'(a) = \frac{1}{3a^{2/3}}.$$

(Here $x^{2/3}$ is defined to be $(x^{1/3})^2$. You may find the factorization of $X^3 - Y^3$ helpful.)

(b) Is f_3 differentiable at 0?

(c) **OPTIONAL!** Let $n \geq 2$ be a natural number, and let f_n be defined by $f_n(x) = x^{1/n}$ (so the domain of f_n is all reals if n is odd, and all non-negative reals if n is even). Prove, directly from the definition, that

$$f_n'(a) = \frac{1}{na^{(n-1)/n}}$$

for all $a \neq 0$ (in the domain of f_n), and that f_n is not differentiable at 0 for any n.

4. Find f' if $f(x) = [x]$ (remember that $[x]$ is the *integer part* of x: the greatest integer less than or equal to x).

5. Imagine a road on which the speed limit is specified at every single point. That is, there is a certain function L such that the speed limit x miles from the beginning of the road is $L(x)$.

   Two cars, A and B, are traveling along the road. A's position at time t is $a(t)$, and B's is $b(t)$.

   (a) What equation expresses the fact that A always travels at the speed limit? (Be careful — question your first answer!)

   (b) Suppose A always goes at the speed limit, and B's position at time t is always A's position at time $t-1$. Show that B is also going at the speed limit at all times.

   (c) Suppose, instead, that B always stays *a constant distance c behind* A. Under what conditions will B always be traveling at the speed limit?

6. (a) Give an example of a function which is continuous at all reals, can be differentiated at all reals, whose derivative is continuous at all reals, but which cannot be differentiated twice times at 0. (**Hint**: The function f defined by $f(x) = |x|$ is not differentiable at 0).

[2] The definition involves a limit; you can assume any facts/theorems we have proven about limits and continuity. This note also applies to the next question.

(b) For each $k \geq 1$, give an example of a function which is continuous at all reals, can be differentiated k times at all reals, and whose kth derivative is continuous at all reals, but which cannot be differentiated $k+1$ times at 0.

7. Recall that $f^{(k)}$ denotes the kth derivative of the function f, and that by convention $f^{(0)}$ means f itself.

   We have

$$(fg)^{(0)} = fg = f^{(0)}g^{(0)},$$
$$(fg)^{(1)} = (fg)' = fg' + f'g = f^{(0)}g^{(1)} + f^{(1)}g^{(0)},$$

   and

$$(fg)^{(2)} = (fg)'' = (fg' + f'g)' = fg'' + 2f'g' + f''g'' = f^{(0)}g^{(2)} + 2f^{(1)}g^{(1)} + f^{(2)}g^{(0)}.$$

   There seems to be a pattern here:

$$(fg)^{(n)} = \sum_{k=0}^{n} (\text{SOME COEFFICIENT DEPENDING ON } n \text{ and } k)\ f^{(k)}g^{(n-k)}.$$

   Find the specific pattern, and prove that is correct for all $n \geq 0$.

8. This question will very likely require using the ε-δ definition of the limit.

   (a) Define f as follows:

$$f(x) = \begin{cases} x^2 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}.$$

   Prove that f is differentiable at 0.

   (b) Let f be a function such that $|f(x)| \leq x^2$ for all x. Prove that f is differentiable at 0.

   (c) **OPTIONAL!** Generalize part b): find a condition on a function g, such that you can prove the following statement:

   > "Let f be a function such that $|f(x)| \leq |g(x)|$ for all x. Then f is differentiable at 0."

   Your condition should be satisfied by the function $g(x) = x^2$.

9. Prove that if f is an even function (one satisfying $f(x) = f(-x)$ for all x) then f' is odd (satisfies $f'(x) = -f'(-x)$ for all x).

10. (a) If $f + g$ is differentiable at a, are f and g necessarily differentiable at a?

    (b) If both fg and f are differentiable at a, what conditions on f imply that g is differentiable at a?

11. **OPTIONAL!** There are many ways to write the identity function I (defined by $I(x) = x$) as a product fg of two differentiable functions — for example, $f(x) = 3x$ and $g(x) = 1/3$. Is it possible to write $I = fg$ where f and g are both differentiable, and satisfy $f(0) = g(0) = 0$?