Math 10850, fall 2019, University of Notre Dame

Notes on first exam — after the fact

October 9, 2019

Stats and grades

People did well on the first four questions: medians/means of 10/9.5, 8.6/7.7, 9.5/8.3 and 8/8. People did less well on the fifth question: median/mean 6/6.2. It's hardly co-incidental that Q5 was on the one topic that you haven't had graded homework on yet[1]. So I decided to weight that question so that it was only worth 50% of the other four[2]. This should explain why the final percentage on your exam is different from your raw percentage.

Here are the stats for the exam, overall:

- First quartile: 91.7
- Median: 82.44
- Mean: 81.35
- Third quartile: 75.7.

These numbers are comparable to the first midterm numbers from 2018 and 2017: Median/mean 84/82.7 for 2018, and 83.3/82.9 for 2017.

I haven't assigned letter grades. But: each of the last two years I've taught this course, everyone who averaged 92% or better over all graded components got an A, and everyone who averaged 88% or better got an A-, and I don't expect things to be any different this year.

1 Specific comments

Here are a few issues that I noted as I graded, that I would like everyone to be aware of.

[1]Evidence that the homework is helping!

[2]Unless that re-weighting brought your score *down* — then I stuck with the original weighting

- **Difference between "equals" and "equivalent"**: you should only use **equals** when comparing two numbers/numerical expressions, or two sets (and so, as a special case, two functions). Reserve **equivalent** for comparing logical expressions. So
  - "$(x+y)(x+z)$ equals $x^2+x(y+z)+yz$", or more succinctly "$(x+y)(x+z) = x^2+x(y+z)+yz$" — GOOD
  - "$(x+y)(x+z)$ is equivalent to $x^2+x(y+z)+yz$" — BAD
  - "$\neg(p \vee q)$ equals/is the same as $(\neg p) \vee (\neg q)$" — BAD
  - "$\neg(p \vee q)$ is equivalent to $(\neg p) \vee (\neg q)$" — GOOD
- **Proving identities**: When trying to establish that two things are equal, say $A = B$, try to avoid starting from $A = B$, and manipulating both sides until you get something that is obviously true, like $0 = 0$. Unless you make it *absolutely clear* in your narrative that each successive line in your verification *implies the previous line* (as opposed to "is implied by"), then there is the real risk that either
  - you will end up deducing a true statement *from* the statement you would like to show is true (which tells you nothing[3]) rather than going the other way around,

  or

  - you will *appear* to be going the wrong way around, which is almost as bad, since a goal of mathematical writing is to *clearly* express mathematical ideas, with as little confusion as possible.[4]

  As an example, to prove that $\binom{n}{k} = \frac{n}{k}\binom{n-1}{k-1}$, you might be tempted to go from

  $$\binom{n}{k} = \frac{n}{k}\binom{n-1}{k-1}$$

  to

  $$\frac{n!}{k!(n-k)!} = \frac{n}{k}\frac{(n-1)!}{(k-1)!((n-1)-(k-1))!}$$

[3]To prove that $-1 = 1$, start with $-1 = 1$ and square both sides to get the true statement $1 = 1$; since this is true, so is the statement we started with. Nonsense, right?

[4]Question: How do you get to Four Winds Field from campus? Answer: "Well, you finish by turning left from Western Ave onto Taylor Street. You get onto Western Ave by turning right from Main Street. You get onto Main Street by continuing south on 933. You get onto 933 by turning left at the end of Dorr Road. You get onto Dorr Road from the parking lot on campus near Rockne Gym."

The directions were perfectly valid, but you'll agree, I hope, that because they were presented *backwards*, they were quite confusing! Unnecessarily so, since I could (should) have said "From the campus parking lot near Rockne Gym, take Dorr Road, turn left onto 933, continue straight onto Main Street, take a right onto Western Ave, thae left onto Taylor Street".

to

$$\frac{n!}{k!(n-k)!} = \frac{n}{k}\frac{(n-1)!}{(k-1)!(n-k)!}$$

to

$$\frac{n!}{k!(n-k)!} = \frac{n}{k}\frac{n!}{(k!(n-k)!}$$

and then conclude that since this last is evidently true, so is the first.

Even if the framing narrative is presented in such a way that the proof is actually valid, it is still bad form, for the reasons described above. Much better would be

$$\begin{aligned}
\binom{n}{k} &= \frac{n!}{k!(n-k)!} \\
&= \frac{n(n-1)!}{k(k-1)!((n-1)-(k-1))!} \\
&= \frac{n}{k}\frac{(n-1)!}{(k-1)!(n-1)-(k-1))!} \\
&= \frac{n}{k}\binom{n-1}{k-1}.
\end{aligned}$$

Every line follows from the previous one directly and clearly, and the flow of the proof goes from where we *are*/what we *know*, to where we *want* to go/what we *want* to know.

- **Clarity of presentation in computational problems**: Question 5 part (b) was computational, but that should not be taken as license to write sloppily. In your future lives, academic or otherwise, you will have occasion to present computations/processes in such a way that you both

  – convey the *answer* clearly

  and

  – convey the *process* that led to the answer clearly.

  Have a look at my write-up of a solution to this problem (on the course webpage): it is clear that I am dividing the domain into three pieces, and on each piece solving $f(x) \geq 0$. The answer clearly conveys the *how* as well as the *what*.

  I was generous in my grading of this part, since it was late in the exam; but having issued this warning, I may be less generous in grading a similar question on the second exam!

- **Proofs *not* by induction**: Not all proofs of $(\forall n \in \mathbb{N})p(n)$ are proofs by induction; indeed, the proof that $\binom{n}{k} = \frac{n}{k}\binom{n-1}{k-1}$ is best done directly.

If you are doing a proof by induction, then you will probably end up *not using* $p(n)$ in your proof that $p(n)$ implies $p(n+1)$; in other words, you will more than likely end up, in the "induction step", giving an *unconditional* proof that $p(n+1)$ is true.

At this point, you should recognize that you are not actually doing a proof by induction, and re-write the proof as a direct proof. While the "proof by induction (that doesn't actually use the principle of induction)" is not *technically* incorrect, it is apt to be very confusing to the reader.[5]

- **Domain of a composition**: The domain of $f \circ h$ is the set of elements x in the domain of h, such that $h(x)$ is in the domain of f. The "rule" for computing $f \circ h$ might seduce you into thinking that the domain is larger, but don't be so seduced!

  On Question 5 part (c), f was the function given by $f(x) = |x^2 - 4| + |x^2 - 1| - 4$ on the domain $[-2, 2]$, and h was the function given by $h(x) = \sqrt{x}$, with no domain specified, so we take the natural domain, the set of non-negative reals. Which non-negative reals have the property that their square root lies between -2 and 2? Those that are between 0 and 4, so the domain of $f \circ h$ is $[0, 4]$.

  One might be tempted to say:

  $$(f \circ h)(x) = |(\sqrt{x})^2 - 4| + |(\sqrt{x})^2 - 1| - 4 = ||x| - 4| + ||x| - 1| - 4,$$

  which makes sense for all reals, so the domain of $f \circ h$ is all reals; but whereas it is true that the (natural) domain of the function given by the rule $x \mapsto ||x|-4|+||x|-1|-4$ is all reals, it's not the case that this is the domain of $f \circ h$; for example

  - $(f \circ h)(-10)$ makes no sense, because $\sqrt{-10}$ doesn't make sense,

  and

  - $(f \circ h)(9)$ makes no sense, because $f(3)$ doesn't make sense (3 being outside the domain of f).

[5] An analogy with giving directions, again: framing a proof as one by induction, when the induction step doesn't actually use the induction hypothesis, is like telling someone that to get from A to B, they should first go to the train station at A and buy a ticket for a train to B, but then, rather than use the train, they should just walk the five minutes down the road from the train station at A to B.