Math 10850, fall 2019, University of Notre Dame

Notes on second exam

November 19, 2019

The key facts

The second midterm will be on

Friday, November 22, in class.

The class runs 50 minutes, but since the next class does not meet until 30 minutes after ours, the exam will run for **1 hour**. There is some flexibility here; I will allow people to continue if they wish to after the hour, but we absolutely must vacate the room by 12.40.

What you need to know

This is course where virtually every new topic builds in some fundamental way on previous topics. In that sense, as an initially preparation for the second midterm, you will need to know everything that you knew for the first midterm. Here is a summary of the topics that we have added since the first midterm[1], followed by a list of theorems that you should know the proofs of, and definitions that you should know correct statements of. There is some overlap between these lists.

- Graphs
  - What is the graph of a function?
  - What are the general shapes of the graphs of some familiar functions:
    - constant?
    - linear?
    - quadratic?

[1]Everything up to the end of class on Friday, November 15 — in the courtse notes, this is:

- Sections 3.7 and 3.8,
- Section 5.5, and
- Sections 6, 7 and 8.

        * general polynomial?
        * circle?
        * $\sin, \cos$ and variants?

- Limits
  - What is the definition of a function tending to a limit?
  - How do you compute basic limits using the formal ε-δ definition?
  - How do you establish that a limit does not exist?
  - What are one-sided limits (from the left/below and right/above)?
- Continuity
  - What is the definition of a function being continuous at a point?
  - What are the different ways in which a function might be not continuous?
  - What conditions ensure that a composition of functions is continuous at a point?
  - What is an example of a function that is defined everywhere but continuous nowhere?
  - What is an example of a function that is defined everywhere, is continuous at all irrationals, but not continuous at any rational?
  - How is the continuity of a function at a point related to the one-sided limits of the function at that point?
  - If a function is continuous at a point, and positive, what can you say about the function in an interval around that point?
  - What does the Intermediate Value Theorem say?[2]
  - What does the Extreme Value Theorem say?[3]
  - Why are the Intermediate Value Theorem and the Extreme Value Theorem false if we work exclusively in the rational numbers?
  - Why are the Intermediate Value Theorem and the Extreme Value Theorem false if we do not assume continuity of functions, on closed intervals?
  - How is the function $x^{1/n}$ defined for natural numbers n?
  - What can you say about odd-degree polynomials, with regards to real roots?
  - What can you say about even-degree polynomials, with regards to boundedness?
- Completeness

[2]There are many variants; the first one introduced is what I think of as *the* IVT.
[3]Remember that is has two parts.

  - What does it mean to say that a set is bounded above, or below?
  - What is an upper bound, or lower bound, for a set?
  - What is a least upper bound for a set, and a greatest lower bound?
  - What is the supremum $\sup A$ of a set A, and the infimum $\inf A$?
  - Which subsets of the reals have suprema and infima (a.k.a least upper bounds and greatest lower bounds)?
  - What is the completeness axiom for the reals?
  - How does the completeness axiom allow a proof of the Intermediate Value Theorem?
  - If a function is continuous at a point, what can you conclude about boundedness around that point?
  - How does the completeness axiom allow a proof that continuous functions on closed intervals are bounded?
  - How does the completeness axiom allow a proof that continuous functions on closed intervals achieve their maxima and minima?
  - How do you show that the natural numbers are unbounded?
  - What is the Archimedean property, and how do you prove it?
  - What does it mean to say that a set is dense in the reals?
  - How do you show that the rationals are dense in the reals, and that the irrationals are dense in the reals?

- Differentiation
  - What does it mean to say that a function is differentiable at a point?
  - How do you calculate the derivative of a function at a point, directly from the definition?
  - What is the linearization of a function at a point?
  - What is the equation of the tangent line to the graph of a function at a point?
  - What is the physical interpretation of the derivative?
  - What can be said about the derivatives of sums and constant multiples of a function/functions?
  - What is the product rule for differentiation?
  - What is the product rule for the product of more than two functions?
  - What is the reciprocal rule for differentiation?
  - What is the quotient rule for differentiation?
  - What is the chain rule for differentiation?

- What is the derivative of a function that is expressed as a composition of more than two functions?
- What are higher derivatives of a function?
- What is the product rule for higher derivatives of the product of two functions?
- What are the derivatives of basic functions such as x^n $x^{1/n}$, $1/x^n$, and the trigonometric functions?
- What is the relationship between differentiability and continuity?

You should be able to prove things like:

- the limit, if it exists, is unique
- the limit of a sum, product or quotient is the sum, product or quotient of the limits[4]
- the squeeze theorem for limits
- continuity of sums, products and quotients of continuous functions
- if g is continuous at a and f is continuous at $g(a)$ then $f \circ g$ is continuous at a
- the Dirichlet function is continuous nowhere
- the Stars over Babylon function (Thomae's function, the popcorn function) is continuous at all irrationals, but not continuous at any rational
- a function that is continuous and positive at a point is positive on some interval around that point
- a function is continuous at a point if and only if it is both left- and right-continuous at that point
- every non-negative number has a unique non-negative nth root, for every even n, and every number has a unique nth root, for every odd n
- the nth root functions are continuous
- the Intermediate Value Theorem
- $\mathbb{N}$ is not bounded above
- the Archimedean property holds
- $1/n$ can be made arbitrarily small

[4]The last only under certain conditions.

- a function that is continuous at a point is bounded in some interval around that point
- a function that is continuous on a closed interval is bounded on the interval (first part of Extreme Value Theorem)
- a function that is continuous on a closed interval achieves its maximum on the interval (second part of Extreme Value Theorem)
- the rationals are dense in the reals
- the irrationals are dense in the reals
- differentiability implies continuity, but not conversely
- the product, reciprocal and quotient rules for differentiation
- the chain rule for differentiation

and you should be able to give formal, correct and complete statements of definitions, concepts, axioms and theorems such as

- limits
- continuity
- limits from the left and limits from the right (from below and from above)
- the Intermediate Value Theorem (in its various forms)
- a function being bounded (above and/or below) on an interval
- a function having a maximum/minimum on an interval
- the Extreme Value Theorem (both parts)
- a set being bounded above or below
- a set having a least upper bound or greatest lower bound
- the completeness axiom
- the Archimedean property
- density of a set in the reals
- the derivative
- the tangent line/slope of a function at a point

Practice questions

This set of problems includes theoretical (proof) questions, and problems. These problems are not intended to be "clones" of the actual exam questions; rather, they are intended to be questions that get you thinking about all the material that may be examined.

The theoretical questions should give you a good idea of the sort of things I will ask you to prove on the exam. Notice that the fine details of the proofs of IVT and EVT are not included. Realistically, these are too long for an in-class exam. Ideally you should understand the details of these proofs, so you can use them later in other contexts; but I won't expect you to reproduce them in the exam.

In some of these questions, it may be unclear whether I'm expecting you to tackle the problem from first principles (essentially, using ε-δ math), or by applying basic theorems/facts we've learned. On the exam, I'll try to make this distinction very clear, question-by-question.

Although many of the questions don't explicitly say this, it is a given that you **justify** all statements you make!

These questions were written quickly and may have some lack of clarity. I will try very hard to make sure that what I expect from you from an exam question is crystal clear. But to help in this, please tell me over the coming days if you find instructions in these questions to be unclear or confusing.

1. Suppose that f approaches a limit L near a, and that also f approaches a limit M near a. Prove that $L = M$. (**Hint**: Argue by contradiction — suppose first that $L \neq M$, and from that derive a contradiction.)

2. Here is a fact that we have used implicitly in class a few times, but that really requires a justification:

   > Let f and g be two functions that are both defined near a. Suppose that there is $\delta > 0$ such that $f(x) = g(x)$ for all $x \in (a-\delta, a) \cup (a, a+\delta)$. Suppose also that $\lim_{x\to a} f(x) = L$ for some number L. Then $\lim_{x\to a} g(x) = L$.

   Using the definition of limit, prove this fact.

3. (a) Suppose that f and g are continuous functions, that $f^2 = g^2$, and that for all x, $f(x) \neq 0$. Prove[5] that EITHER

   $f(x) = g(x)$ for all x,

   OR

   $f(x) = -g(x)$ for all x.[6]

[5] Not necessarily directly from the ε-δ definition of the limit; here you can (and should) make use of the various theorems about continuity that we have proven.

[6] Note that this is *quite* different from

(b) Show that if we drop the assumption $(\forall x)(f(x) \neq 0)$, the conclusion above is not necessarily true[7].

4. Suppose that f and g are both continuous at a.

   (a) Prove, using the ε-δ definition of continuity, that $f+g$ is continuous at a.

   (b) Prove, using the ε-δ definition of continuity, that fg is continuous at a.

5. (a) Suppose that f is continuous at a. Show, *directly from the ε-δ definition of continuity*, that $|f|$ is continuous at a.

   (b) Give a quick explanation for why continuity of f at a implies continuity of $|f|$ at a, based not on the definition (as you did in part (a)), but rather on some general results that we have proven. (Start by arguing that the absolute value function is continuous everywhere, again based not on definitions but on general results).

   (c) If $|f|$ is continuous, must f be?

6. Let f, g and h be three functions satisfying $f(x) \leq g(x) \leq h(x)$ for all x, and satisfying $f'(a) = h'(a)$.

   (a) Suppose that $f(a) = h(a)$. Prove (most likely directly using the definition of differentiability) that g is differentiable at a, and that $f'(a) = g'(a) = h'(a)$.

   (b) Show (by way of an example) that g is not necessarily differentiable at a if we drop the condition that $f(a) = h(a)$.

7. For each of the following functions,

   - prove, directly from the ε-δ definition of continuity, that the function is continuous at $a = 5$,
   - calculate the derivative at the point $a = 5$, directly from the definition[8],
   - compute the linearization $L_{f,5}(x)$ of the function,
   - use the linearization to estimate the value of $f(6)$, and
   - (using a calculator) calculate the percentage error in using $L_{f,5}(6)$ as an approximation for $f(6)$.

   (a) $f(x) = 1/\sqrt{x-1}$

"for all x, either $f(x) = g(x)$ or $f(x) = -g(x)$". $(\star)$

You should be able to prove $(\star)$ quite easily. If you cannot see why $(\star)$ is different from the statement the question is asking you to prove, come ask me about it, or ask in the bunker, or ask Sarah, or ask a friend!

[7] This will probably require a specific example.

[8] But here you need not use the ε-δ formalism; you can use any and all facts we have established about continuity and limits.

(b) $f(x) = x^2 + (1/x^2)$.

8. Suppose that f is a function defined on all reals such that $f(x) \leq f(y)$ whenever $x < y$. Let a be a number, and let $A = \{f(x) : x < a\}$.

    (a) Show that A is not empty, and that it is bounded above.

    (b) Show that $\lim_{x \to a^-} f(x)$ exists.

9. In class we used the binomial theorem to prove that for $n \in \mathbb{N}$, if $f(x) = x^n$ then $f'(x) = nx^{n-1}$. Give an alternate proof of this fact, using the product rule and induction.

10. Let $f : \mathbb{R} \to \mathbb{R}$ be an odd function ($f(x) = -f(-x)$ for all x).

    (a) Using the definition of the derivative, show that for $x > 0$, f is differentiable at x if and only if f is differentiable at $-x$.

    (b) Suppose that f is differentiable everywhere. Show that f' is an even function ($f'(-x) = f'(x)$). (**Hint**: this is much easier if it is viewed as a question about the chain rule, rather than about the definition of the derivative.)

11. Two problems involving the chain rule:

    (a) A circular object is increasing in size in some unspecified manner, but it is known that when the radius is 6 meters, the rate of change of the radius is 4 meters/second^2. Find the rate of change of the area when the radius is 6 meters.

    (b) The area between two varying concentric circles is at all times 9π cm^2. The rate of change of the area of the larger circle is 10π cm^2 per second. How fast is the circumference of the smaller circle changing when it has area 16π cm^2?

12. Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x} & \text{if } xneq0 \\ 0 & \text{if } x = 0. \end{cases}$$

    (a) Show that f is differentiable at all reals, and find its derivative.

    (b) Show that f is not twice differentiable at 0.

    (c) Suppose that h and k are two functions, defined at all reals, satisfying

    - $h(0) = 3$ and $h'(x) = \sin^2(\sin(x + 1))$
    - $k(0) = 0$ and $k'(x) = f(x + 1)$.

    Find

    i. $(f \circ h)'(0)$

    ii. $(k \circ f)'(0)$

    iii. $\alpha'(x^2)$ where $\alpha(x) = h(x^2)$.