Math 10850, Honors Calculus 1

Quiz 4, Thursday September 26

Solutions

1. State the principle of mathematical induction *clearly* and *completely*. I'll get things started for you: "Let $p(n)$ be a predicate, where the universe of discourse for n is $\mathbb{N}$ ...

   Solution: If $p(1)$ is true, and if, for all $n \in \mathbb{N}$, $p(n)$ implies $p(n+1)$, then $p(n)$ is true for all $n \in \mathbb{N}$.

2. Let k be a fixed natural number. Prove (carefully, and with a neat & clear layout) that for all $n \geq k$,

$$\binom{k}{k} + \binom{k+1}{k} + \cdots + \binom{n}{k} = \binom{n+1}{k+1}.$$

   Solution: We prove the identity by induction on n.

   Base case, $n = k$: the identity asserts that $\binom{k}{k} = \binom{k+1}{k+1}$, which is true because both sides equal 1.

   Induction step: We assume that for some $n \geq k$, it holds that

$$\binom{k}{k} + \binom{k+1}{k} + \cdots + \binom{n}{k} = \binom{n+1}{k+1},$$

   and we wish to use this to deduce that

$$\binom{k}{k} + \binom{k+1}{k} + \cdots + \binom{n}{k} + \binom{n+1}{k} = \binom{n+2}{k+1}.$$

   We have

$$\begin{aligned} \binom{k}{k} + \binom{k+1}{k} + \cdots + \binom{n}{k} + \binom{n+1}{k} &= \left(\binom{k}{k} + \binom{k+1}{k} + \cdots + \binom{n}{k}\right) + \binom{n+1}{k} \\ &= \binom{n+1}{k+1} + \binom{n+1}{k} \quad \text{(induction hypothesis)} \\ &= \binom{n+2}{k+1} \quad \text{(Pascal's identity)}, \end{aligned}$$

   as required.

   By induction, the identity is true for all $n \geq k$.