Math 10850, fall 2019

Second midterm exam, Friday November 22

Solutions and comments

Stats and grades

Here are the median percentages for the four questions: 89.3, 91.7, 84.6 and 77.3. Clearly, the last question was the one the people found the hardest. But really, the hardest question was the bonus question. Everyone got some credit, but only **three** people obtained the correct answer and got full credit.

Here are the stats for the exam, overall:

- First quartile: 89
- Median: 86
- Mean: ≈ 80.6
- Third quartile: 68.

These numbers are better than the second midterm numbers from 2018 and 2017: Median/mean 80/82 for 2018, and 82/83 for 2017.

I haven't assigned letter grades. But: each of the last two years I've taught this course, everyone who averaged 92% or better over all graded components got an A, and everyone who averaged 88% or better got an A-, and I don't expect things to be any different this year.

Solutions, with comments

1. (a) (3 pts) Give the (precise, ε-δ) definition of "f approaches the limit L near a".

   Solution: This means that for all $\varepsilon > 0$ there is a $\delta > 0$ such that for all x, if $0 < |x - a| < \delta$ then $|f(x) - L| < \varepsilon$. (There are other ways of expressing this, of course, which are equally correct.)

   Comment: The definition of "f approaches the limit L near a" requires introducing three numbers (ε, δ and x). All three of these need to be quantified over, otherwise the definition will not be a statement, but rather will be a predicate (depending on the unquantified variable or variables). The variable that some people forgot to quantify over was x.

   (b) (5 pts) Calculate $\lim_{x\to 2} \frac{3x}{1+x}$ *directly from the definition of limit.*

   Solution: Motivated by the fact that $3(2)/(1+2) = 2$, we conjecture that the limit is 2.

Claim: $\lim_{x\to 2} \frac{3x}{1+x} = 2$.

Proof: Let $\varepsilon > 0$ be given. We want to show that there is a $\delta > 0$ such that whenever x is such that $0 < |x-2| < \delta$, we have $\left|\frac{3x}{1+x} - 2\right| < \varepsilon$.
Now

$$\begin{aligned} \left|\frac{3x}{1+x} - 2\right| &= \left|\frac{3x}{1+x} - \frac{2+2x}{1+x}\right| \\ &= \left|\frac{3x-(2+2x)}{1+x}\right| \\ &= \left|\frac{x-2}{1+x}\right| \\ &= \frac{|x-2|}{|1+x|}, \end{aligned}$$

so to show $\left|\frac{3x}{1+x} - 2\right| < \varepsilon$ it is enough to show $\frac{|x-2|}{|1+x|} < \varepsilon$.
If $\delta \leq 1$ then $0 < |x-2| < \delta$ implies that $-1 < x-2 < 1$ which implies $2 < x+1 < 4$, so $|x+1| > 2$ and $\frac{|x-2|}{|1+x|} < |x-2|/2$.
If *also* $\delta \leq 2\varepsilon$ then $0 < |x-2| < \delta$ further implies $\frac{|x-2|}{|1+x|} < (2\varepsilon)/2 = \varepsilon$.
It follows that if $\delta \leq \min\{1, 2\varepsilon\}$ then $0 < |x-2| < \delta$ implies $\left|\frac{3x}{1+x} - 2\right| < \varepsilon$.
Since $\varepsilon > 0$ was arbitrary, and $\min\{1, 2\varepsilon/3\} > 0$ (so there is actually a $\delta > 0$ with $\delta \leq \min\{1, \varepsilon/3\}$, this proves that $\lim_{x\to 2} \frac{3x}{1+x} = 2$.

Comments:

- In the definition, the existential quantification of δ comes *before* any mention of x, so δ is *only* allowed to depend on ε (some people had δ depending on x as well as ε).
- Notice that at no point in this proof did I write down a statement, equation or inequality, without relating it to a previous statement, equation or inequality (for example, by saying that the statement follows from a previous one, or that it is a statement that is equivalent to, or would imply, the thing that we want to prove).
  At no point in **any** proof should you write down a statement, without given a clear indication of its function in the proof. Without such an indication, the person looking at your proof needs to do some mind-reading to figure out what you had intended.
- On a related point, note also that in the string of equalities where I established that $\left|\frac{3x}{1+x} - 2\right| < \varepsilon$ is the same as $\frac{|x-2|}{|1+x|} < \varepsilon$, I actually wrote down a string of inequalities. I did not write:

$$\left|\frac{3x}{1+x} - 2\right|$$
$$\left|\frac{3x}{1+x} - \frac{2+2x}{1+x}\right|$$
$$\left|\frac{3x-(2+2x)}{1+x}\right|$$
$$\left|\frac{x-2}{1+x}\right|$$
$$\frac{|x-2|}{|1+x|},$$

(in a way that leaves the reader wondering what these all have to do with one another), nor did I write

$$\begin{gathered}\left|\frac{3x}{1+x} - 2\right| < \varepsilon \\ \left|\frac{3x}{1+x} - \frac{2+2x}{1+x}\right| < \varepsilon \\ \left|\frac{3x-(2+2x)}{1+x}\right| < \varepsilon \\ \left|\frac{x-2}{1+x}\right| < \varepsilon \\ \frac{|x-2|}{|1+x|} < \varepsilon,\end{gathered}$$

which has the same issue. Finally, I did not write

$$\begin{aligned} & \left|\frac{3x}{1+x} - 2\right| < \varepsilon \\ \implies & \left|\frac{3x}{1+x} - \frac{2+2x}{1+x}\right| < \varepsilon \\ \implies & \left|\frac{3x-(2+2x)}{1+x}\right| < \varepsilon \\ \implies & \left|\frac{x-2}{1+x}\right| < \varepsilon \\ \implies & \frac{|x-2|}{|1+x|} < \varepsilon.\end{aligned}$$

This is a correct sequence of implications, but a useless one. Ultimately I want to show that something (δ small) forces $\left|\frac{3x}{1+x} - 1\right| < \varepsilon$, so I want to establish something that **implies** this conclusion (and that I can prove), not something that **is implied by the conclusion** (and that I can prove) — if I want to deduce A, and I know that $A \implies B$, then proving B tells me precisely nothing. If, on the other hand, I want to deduce A, and I know that $B \implies A$, then proving B tells me everything. So, the correct phrasing for the string above would have been either:

$$\begin{aligned} & \left|\frac{3x}{1+x} - 2\right| < \varepsilon \\ \text{is implied by} \quad & \left|\frac{3x}{1+x} - \frac{2+2x}{1+x}\right| < \varepsilon \\ \text{which is implied by} \quad & \left|\frac{3x-(2+2x)}{1+x}\right| < \varepsilon \\ \text{which is implied by} \quad & \left|\frac{x-2}{1+x}\right| < \varepsilon \\ \text{which is implied by} \quad & \frac{|x-2|}{|1+x|} < \varepsilon,\end{aligned}$$

or,

$$\begin{aligned} & \left|\frac{3x}{1+x} - 2\right| < \varepsilon \\ \iff & \left|\frac{3x}{1+x} - \frac{2+2x}{1+x}\right| < \varepsilon \\ \iff & \left|\frac{3x-(2+2x)}{1+x}\right| < \varepsilon \\ \iff & \left|\frac{x-2}{1+x}\right| < \varepsilon \\ \iff & \frac{|x-2|}{|1+x|} < \varepsilon.\end{aligned}$$

(c) (4 pts) Let $f(x) = \frac{3x}{1+x}$. Compute $f'(x)$ for x in the domain of f, directly from the definition of derivative. (Here you may assume any reasonable facts that we have proven about limits and continuous functions).

Solution: As long as x is not -1 (i.e., as long as x is in the domain of f), and as long as h is not 0, we have

$$\begin{aligned}
\frac{f(x+h)-f(x)}{h} &= \frac{\frac{3(x+h)}{1+(x+h)} - \frac{3x}{1+x}}{h} \\
&= \frac{3(x+h)(1+x) - 3x(1+(x+h))}{h(1+(x+h))(1+x)} \\
&= \frac{3x + 3x^2 + 3h + 3hx - 3x - 3x^2 - 3hx}{h(1+(x+h))(1+x)} \\
&= \frac{3h}{h(1+(x+h))(1+x)} \\
&= \frac{3}{(1+(x+h))(1+x)}.
\end{aligned}$$

As long as $x \neq -1$ (which it is never), the final expression above is a rational function of h that is continuous at $h = 0$, so we can evaluate the limit of the expression as $h \to 0$ by direct evaluation. This gives

$$f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h \to 0} \frac{3}{(1+(x+h))(1+x)} = \frac{3}{(1+x)^2}.$$

(d) (2 pts) Compute $(f \circ f)'(2)$. (**Hint**: you shouldn't need to compute the composition $f \circ f$ for this.)

Solution: By the chain rule, and using $f(2) = 2$ (direct evaluation) and $f'(2) = 1/3$ (from part (c)), we have

$$(f \circ f)'(2) = f'(f(2))f'(2) = f'(2)f'(2) = (1/3)(1/3) = 1/9.$$

2. In this question you may *not* assume any reasonable facts that we have proven about limits and continuous functions — I'm looking for ε-δ proofs.

(a) (6 pts) Suppose that f is continuous at 0, and that $f(0) > 0$. Prove that there is a $\delta > 0$ such that $f(x) > 2f(0)/3$ for all x in the interval $(-\delta, \delta)$.

Solution: By the definition of continuity of f at 0, applied at $\varepsilon = f(0)/3$ (note this is > 0), we have that there is a $\delta > 0$ such that for all x in the interval $(-\delta, \delta)$ (i.e., for all x satisfying $|x - 0| < \delta$), we have $|f(x) - f(0)| < f(0)/3$. But this says that $2f(0)/3 < f(x) < 4f(0)/3$, which in particular implies $f(x) > 2f(0)/3$.

(b) (6 pts) Suppose that f and g are functions that are both continuous at a. Prove that $f + g$ is continuous at a.

Solution: Let $\varepsilon > 0$ be given.

Since f is continuous at a, there is $\delta_1 > 0$ such that $|x-a| < \delta_1$ implies $|f(x)-f(a)| < \varepsilon/2$, and by the same token since g is continuous at a, there is $\delta_2 > 0$ such that $|x - a| < \delta_2$ implies $|g(x) - g(a)| < \varepsilon/2$.

Take $\delta = \min\{\delta_1, \delta_2\}$. If $|x-a| < \delta$ then both $|x-a| < \delta_1$ and $|x-a| < \delta_2$, and so both $|f(x) - f(a)| < \varepsilon/2$ and $|g(x) - g(a)| < \varepsilon/2$, and so

$$\begin{aligned}|(f+g)(x) - (f+g)(a)| &= |f(x) + g(x) - f(a) - g(a)| \\ &= |f(x) - f(a) + g(x) - g(a)| \\ &\leq |f(x) - f(a)| + |g(x) - g(a)| \quad \text{(triangle inequality)} \\ &< \varepsilon/2 + \varepsilon/2 \\ &= \varepsilon\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, this shows that $f+g$ approaches limit $f(a)+g(a) = (f+g)(a)$ near a, i.e., that $f+g$ is continuous at a.

Comments:

- This question explicitly forbade use of general theorems, such as the sum theorem for limits.
- In this prove, the key point was to apply the definition of continuity to get $f(x)$ within $\varepsilon/2$ of $f(a)$ (and $g(x)$ within $\varepsilon/2$ of $g(a)$). One's inclination initially is to write this as:

  Applying the definition of continuity with $\varepsilon = \varepsilon/2$ ...

  Anyone should know what this means, but formally it is nonsense, since the expression "$\varepsilon = \varepsilon/2$" fixes $\varepsilon = 0$. It's much better to sidestep this phase, perhaps by writing

  Given $\varepsilon > 0$. Since $\varepsilon/2 > 0$, from the definition of continuity there is $\delta_1 > 0$ such that for all $x \in (a - \delta_1, a + \delta_1)$ it holds that $|f(x) - f(a)| < \varepsilon/2$. ...

3. (a) (3 pts) State the completeness axiom for the reals.

   Solution: Every non-empty set of reals that has an upper bound, has a least upper bound.

   (b) (3 pts) Show that if a and b are both least upper bounds of a set A, then $a = b$.

   Solution: Since a and b are both least upper bounds of A, they are both also upper bounds.

   Since a is a least upper bound of A, it is at least as small as every other upper bound. In particular that means that $a \leq b$.

   But also since b is a least upper bound of A, it is at least as small as every other upper bound. In particular that means that $b \leq a$.

   Since $a \leq b$ and $b \leq a$, we have $a = b$.

   (c) Let $A = \left\{\frac{n}{n+1} : n \in \mathbb{N}\right\}$.

   i. (3 pts) Show that A is non-empty and bounded above.

   Solution: A is non-empty because, for example, $11/12 \in A$ ($n = 11$). It is bounded above, for example, by 1: we have $n \leq n+1$ for all $n \in \mathbb{N}$, and so $n/(n+1) \leq 1$ for all $n \in \mathbb{N}$.

ii. (4 pts) What is the least upper bound of A? (*Carefully* justify!)

Solution: We claim that 1 is the least upper bound of A. It is an upper bound, as shown in the solution to the last part of the question. We argue that it the least upper bound, by contradiction.
Suppose $\alpha < 1$ is an upper bound for A.Because α is an upper bound for A, we have $n/(n+1) \leq \alpha$ for all $n \in \mathbb{N}$, so $n \leq \alpha + \alpha n$, so $(1-\alpha)n \leq \alpha$, so $n \leq \alpha/(1-\alpha)$ for all $n \in \mathbb{N}$ (note $1-\alpha > 0$ since $\alpha < 1$).
But this says that $\mathbb{N}$ is bounded above, which it is not (as we have proven in class). This contradiction shows that A has no upper bounds less than 1, so 1 is the least upper bound.

Comment: Some people also added an argument that it is not possible for $\sup A > 1$, saying that if $\sup A > 1$ then $1 \in A$, which is not possible since there is no $n \in \mathbb{N}$ with $n = n+1$. This doesn't work: just because the supremum of a set is greater than some number a, doesn't force a to be in the set (that would —em only be true if the set were an interval).
The correct argument that $\sup A \not> 1$ is simply the observation that 1 is *an* upper bound, so the least upper bound must be at least as small as 1.

4. (a) (3 pts) Suppose that $f(x) = xg(x)$ where g is differentiable at 0. Find $f'(0)$ in terms of g.

Solution: By the product rule, at a point x where both x and $g(x)$ are differentiable, we have
$$f'(x) = xg'(x) + (x)'g(x) = xg'(x) + g(x).$$
Applying at $x = 0$ we get $f'(0) = 0g'(0) + g(0) = g(0)$.

(b) (5 pts) Suppose that $f(x) = xg(x)$, where g is some function that is continuous at 0. Prove that f is differentiable at 0, and find $f'(0)$ in terms of g.

Solution: We **cannot** use the product rule for differentiation here since g is not known to be differentiable at 0 (I'll elaborate on this below), so we have to examine the limit that defines the derivative of f at 0 (if it exists). We have
$$\begin{aligned}\frac{f(0+h) - f(0)}{h} &= \frac{(0+h)g(0+h) - 0g(0)}{h} \\ &= \frac{hg(h)}{h} \\ &= g(h) \quad \text{as long as } h \neq 0.\end{aligned}$$
We know that $\lim_{h\to 0} g(h) = g(0)$ (by continuity of g at 0), and so
$$\lim_{h\to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0} g(h) = g(0).$$
It follows that $f'(0)$ exists and equals $g(0)$.

Comment: One approach a few people took was to say that $f'(x) = xg'(x) + g(x)$ via the product rule, and then note that although this wasn't strictly speaking allowed (since $g'(0)$ wasn't known to exist), it was ok at $x = 0$ since the contribution from the term involving $g'(0)$ was being forced to be 0 by the presence of the x multiplying $g'(x)$; and then what was left was the (well-defined) $g(0)$.

I marked down for this, because it an instance of a **invalid** argument, even if it leads to a valid conclusion.

To illustrate the danger of using a formula at a point where it isn't defined, but then infering something from the result because the problematic part of the formula is being multiplied by 0, here is an example:

Claim: Suppose that f and g are functions defined for all reals (so $f \circ g$ is defined for all reals), and that g is differentiable at 0, with derivative 0, but that f is not necessarily differentiable at $g(0)$. Then, even still, $f \circ g$ is differentiable at 0, with derivative 0.

Proof: We apply the chain rule:

$$(f \circ g)'(0) = f'(g(0))g'(0).$$

While f is not necessarily differentiable at $g(0)$, that's ok, because the expression "$f'(g(0))$" on the right-hand side of the chain rule is being multiplied by $g'(0)$, which is 0, so the whole product will always be 0. Hence, $(f \circ g)'(0) = 0$, as claimed.

Unfortunately, this result is **false**. Let f be the cubed root function $f(x) = x^{1/3}$, and g the cube function $g(x) = x^3$. Both are defined for all reals, and g is differentiable at 0, with derivative 0. But it is **not** the case that $(f \circ g)'(0) = 0$. In fact, since $(f \circ g)(x) = x$ for all x, we have $(f \circ g)'(0) = 1$. The problem is that f is not differentiable at 0 (which is what $g(0)$ is in this case) (it has an infinite slope at that point), so no valid conclusion can be drawn from the chain rule.

Bottom line: In your arguments, you should only ever use the *conclusions* of theorems when you are certain that the *hypothesis* of those theorems are valid.

(c) (3 points) Suppose that f is differentiable at 0 and that $f(0) = 0$. Prove that $f(x) = xg(x)$ for some function g that is continuous at 0.

Solution: Away from 0, the obvious choice for g (the *only* choice for g) is

$$g(x) = \frac{f(x)}{x}.$$

The only choice for $g(0)$ that makes g continuous at 0 is

$$g(0) = \lim_{x \to 0} \frac{f(x)}{x},$$

if this limit exists. But, using that $f(0) = 0$ and that $f'(0)$ exists, we get that

$$f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x} = \lim_{x \to 0} \frac{f(x)}{x}.$$

So $\lim_{x \to 0} f(x)/x$ exists and equals $f'(0)$.

If we define

$$g(x) = \begin{cases} \frac{f(x)}{x} & \text{if } x \neq 0 \\ f'(0) & \text{if } x = 0 \end{cases}$$

then we have just argued that g is continuous at 0. But also, clearly $f(x) = xg(x)$ for $x \neq 0$, while also $f(0) = 0$ and $0g(0) = 0f'(0) = 0$, so $f(x) = xg(x)$ for $x = 0$ also.

Thus the function g that we have defined satisfies all the required conditions.

Comment: Most people handled this one poorly, by treating g as some given of the problem. But it is *not* a given — the question is specifically asking you to *produce* a function g that makes $f(x) = xg(x)$. You cannot make *any* arguments about the function g, until you have explicitly said what you think g to be.

Bonus question: In class, we discussed the following process:

> Start with a square sheet of paper of side length 1. Cut out four equal-area squares from each of the four corners, and fold up the resulting tabs to form a box with an open top.

We considered (and solved) the question: what is the maximum volume of a box that can be formed by this process?

Here's an earth-friendly refinement:

> The process described above leaves four smaller squares of paper unused. Each of those can be turned into a box with an open top, by the same process as described above. This leaves sixteen even smaller squares of paper unused, each of which can be turned into a box with an open top ... and so on, *ad infinitum*.

What is the maximum *sum* of the volumes of the boxes that can be achieved by this less-wasteful process? Give your final answer as a decimal, to 5 decimal places.

Solution: The maximum sum is around 0.0755605 (and not 0.075471 — see below).

At first this seems like a problem that involves infinitely many variables; one has to choose the side length x_1 of the first four squares to cut out; then the side length x_2 of the four squares to cut out of each of the four left-over squares of side length x_1, and so on (note that the four squares of side length x_1 are identical, so in an optimal scheme all four will be treated in the same way, so there is one variable at the second step, rather than four).

But it fact, this can be turned into a problem involving a single variable. Suppose that in an optimal scheme, the four squares cut out of the initial square have side length x_1. How do we continue with the process, on each of the four squares of side length x_1? Changing units, so that a distance of x_1 now becomes "1", we have four identical copies of the original problem, and for each one of those, there is an optimal scheme that begins by cutting out four squares of side length x_1 in the new units, so x_1^2 in the old units.

Repeating this argument, we see that there is a single number x such that an optimal scheme starts with cutting out four squares of side length x; then cutting out four squares of side length x^2 from each of the four squares of side length x; then cutting out four squares of side length x^3 from each of the sixteen squares of side length x^2; then cutting out four squares of side length x^4 from each of the sixty-four squares of side length x^3; and so on. This leads to a total volume of

$$x(1-2x)^2 + 4x^2(x-2x^2)^2 + 16x^3(x^2-2x^3)^2 + 64x^4(x^3-2x^4)^2 + \ldots$$

or

$$(x-4x^2+4x^3) + 4(x^4-4x^5+4x^6) + 16(x^7-4x^8+4x^9) + 64(x^{10}-4x^{11}+4x^{12}) + \ldots$$

or

$$x(1+4x^3+16x^6+64x^9+\ldots) - 4x^2(1+4x^3+16x^6+64x^9+\ldots) + 4x^3(1+4x^3+16x^6+64x^9+\ldots)$$

or

$$(x - 4x^2 + 4x^3)(1 + 4x^3 + (4x^3)^2 + (4x^3)^3 + \dots)$$

or (using the standard geometric series summation formula)

$$\frac{x - 4x^2 + 4x^3}{1 - 4x^3}.$$

So the question is: what choice of x maximizes $(x - 4x^2 + 4x^3)/(1 - 4x^3)$ as x runs over the interval $[0, 1/2]$? At $x = 0$ and $x = 1/2$ the function to be maximized is (of course) 0, and it is continuous and differentiable for all values on the interval (the only potential problem x is that x such that $1 - 4x^3 = 0$, which is $x \approx 0.63$).

The derivative of $(x - 4x^2 + 4x^3)/(1 - 4x^3)$ is

$$\frac{1 - 8x + 12x^2 + 8x^3 - 16x^4}{(1 - 4x^3)^2}.$$

We cannot compute the zeros of the numerator exactly; Wolfram Alpha says that the only zero in $[0, 1/2]$ is $x^\star \approx 0.173648$, and at this point, the volume function $(x - 4x^2 + 4x^3)/(1 - 4x^3)$ is approximately 0.0755605. This is the maximum attainable volume.

Comment: Note that the maximum volume attainable by using just one step of this process is (as we saw in class) attained by cutting out four squares of side length 1/6, to get a volume of

$$\frac{1}{6}\left(1 - \frac{2}{6}\right)^2 = \frac{2}{27} \approx 0.0740740.$$

If we use the scheme "from remaining squares, always cut out four squares of side length equal to 1/6 of the side length of the remaining square", then the total volume we get is the value of $(x - 4x^2 + 4x^3)/(1 - 4x^3)$ at $x = 1/6$, which is

$$\frac{4}{53} \approx 0.0754717.$$

So we do better (only slightly better — about one tenth of one percent better — but still better) by increasing 1/6 to around 0.173648.