Math 10860, Honors Calculus 2

Wallis' formula for π

March 13, 2018

In honor of π day, March 14, I talked in class about Wallis' product formula for π. These notes fill in the details for those who were absent. They also expand a little on the topic I incoherently extemporized on at the end of class — namely, the connection between Wallis' formula and the binomial coefficients, and why I might care about this connection.

We begin by defining, for integers $n \geq 0$, $S_n := \int_0^{\pi/2} \sin^n x dx$. We have

$$S_0 = \frac{\pi}{2}, \quad S_1 = \int_0^{\pi/2} \sin x dx = 1,$$

and for $n \geq 2$ we get from integration by parts (taking $u = \sin^{n-1} x$ and $dv = \sin x dx$, so that $du = (n-1)\sin^{n-2} x \cos x dx$ and $v = -\cos x$) that

$$\begin{aligned} S_n &= (\sin^{n-1} x)(-\cos x)|_{x=0}^{\pi/2} - \int_0^{\pi/2} -(n-1)\cos x \sin^{n-2} x \cos x dx \\ &= (n-1)\int_0^{\pi/2} \cos^2 x \sin^{n-2} x dx \\ &= (n-1)\int_0^{\pi/2} (1-\sin^2 x)\sin^{n-2} x dx \\ &= (n-1)S_{n-2} - (n-1)S_n, \end{aligned}$$

which leads to the recurrence relation

$$S_n = \frac{n-1}{n} S_{n-2} \quad \text{for } n \geq 2.$$

Iterating the recurrence relation until the initial conditions are reached, we get that

$$S_{2n} = \left(\frac{2n-1}{2n}\right)\left(\frac{2n-3}{2n-2}\right)\cdots\left(\frac{3}{4}\right)\left(\frac{1}{2}\right)\frac{\pi}{2}$$

and

$$S_{2n+1} = \left(\frac{2n}{2n+1}\right)\left(\frac{2n-2}{2n-1}\right)\cdots\left(\frac{4}{5}\right)\left(\frac{2}{3}\right)1.$$

Taking the ratio of these two identities and rearranging yields

$$\frac{\pi}{2} = \left(\frac{2}{1}\right)\left(\frac{2}{3}\right)\left(\frac{4}{3}\right)\left(\frac{4}{5}\right)\cdots\left(\frac{2n}{2n-1}\right)\left(\frac{2n}{2n+1}\right)\frac{S_{2n}}{S_{2n+1}}.$$

Now since $0 \le \sin x \le 1$ on $[0, \pi/2]$ we have also

$$0 \le \sin^{2n+1} x \le \sin^{2n} x \le \sin^{2n-1} x,$$

and so, integrating and using the recurrence relation, we get

$$0 \le S_{2n+1} \le S_{2n} \le S_{2n-1} = \frac{2n+1}{2n} S_{2n+1}$$

and so

$$1 \le \frac{S_{2n}}{S_{2n+1}} \le 1 + \frac{1}{2n}.$$

This says that by choosing n large enough, the ratio S_{2n}/S_{2n+1} can be made arbitrarily close to 1, and so the product

$$\left(\frac{2}{1}\right)\left(\frac{2}{3}\right)\left(\frac{4}{3}\right)\left(\frac{4}{5}\right)\cdots\left(\frac{2n}{2n-1}\right)\left(\frac{2n}{2n+1}\right)$$

can be made arbitrarily close to $\pi/2$ by choosing n large enough. This fact as usually expressed by saying that $\pi/2$ can be described by an "infinite product":

$$\frac{\pi}{2} = \left(\frac{2}{1}\right)\left(\frac{2}{3}\right)\left(\frac{4}{3}\right)\left(\frac{4}{5}\right)\left(\frac{6}{5}\right)\left(\frac{6}{7}\right)\cdots.$$

This infinite product was probably first written down by John Wallis in 1655. Wallis' other claim to fame is that he was probably the first mathematician to use the symbol "∞" for infinity.

Note that Wallis' formula is not a particularly good way to actually estimate π; because we have $1 \le S_{2n}/S_{2n+1} \le 1 + 1/2n$, it turns out that to get an estimate of π correct to k decimal places, we need to take $n \approx 10^k$. This is similar to the rate of convergence of the approximation based on $\arctan 1$.

Wallis' formula can be used to estimate the binomial coefficient $\binom{2n}{n}$. Indeed,

$$\begin{aligned}\binom{2n}{n} &= \frac{(2n)(2n-1)(2n-2)\cdots(3)(2)(1)}{(n)(n-1)\cdots(2)(1)(n)(n-1)\cdots(2)(1)} \\ &= 2^n\frac{(2n-1)(2n-3)\cdots(3)(1)}{(n)(n-1)\cdots(2)(1)} \\ &= 2^{2n}\frac{(2n-1)(2n-3)\cdots(3)(1)}{(2n)(2n-2)\cdots(4)(2)} \\ &= \frac{2^{2n}}{\sqrt{2n+1}}\sqrt{\frac{(2n+1)(2n-1)(2n-1)(2n-3)(2n-3)\cdots(3)(3)(1)}{(2n)(2n)(2n-2)(2n-2)\cdots(4)(4)(2)(2)}}\end{aligned}$$

and so

$$\frac{\sqrt{n}\binom{2n}{n}}{2^{2n}} = \sqrt{\frac{n}{2n+1}}\sqrt{\frac{(2n+1)(2n-1)(2n-1)(2n-3)(2n-3)\cdots(3)(3)(1)}{(2n)(2n)(2n-2)(2n-2)\cdots(4)(4)(2)(2)}}$$

For large enough n, $\sqrt{n/(2n+1)}$ can be made arbitrarily close to $1/\sqrt{2}$, and the other term on the right-hand side above can (by Wallis' formula) be made arbitrarily close to $\sqrt{2/\pi}$, so the whole right-hand side can be made arbitrarily close to $\sqrt{1/\pi}$. In other words,

$$\lim_{n\to\infty} \frac{\sqrt{n}\binom{2n}{n}}{2^{2n}} = \frac{1}{\sqrt{\pi}}.$$

(Note that this is not a very helpful limit: at $n = 10,000$ the expression $\sqrt{n}\binom{2n}{n}/2^{2n}$ evaluates to around 0.564183, whereas $1/\sqrt{\pi} \approx 0.564189$).

This limit is usually written

$$\frac{\binom{2n}{n}}{2^{2n}} \sim \frac{1}{\sqrt{n\pi}} \quad \text{as } n \to \infty;$$

here I am introducing the symbol "$\sim$", read as "asymptotic to", which is defined as follows:

$$f(n) \sim g(n) \quad \text{as } n \to \infty$$

if $\lim_{n\to\infty} f(n)/g(n) = 1$. The sense is that f and g grow at essentially the same rate as n grows. Note that this does *not* say that f and g get closer to one another absolutely as n grows; for example $n^2 \sim n^2 + n$ as $n \to \infty$, but the difference between the two sides goes to infinity too. It's the *relative* (or proportional) difference that gets smaller.

This estimate for $\binom{2n}{n}$ has a connection to probability. If a fair coin is tossed $2n$ times, then the probability that it comes up heads exactly k times is $\binom{2n}{k}/2^{2n}$. This quantity is at its largest when $n = k$ (some easy algebra), at which point it takes value very close to $1/\sqrt{n\pi}$ (as we have just discovered).

Some easy algebra also suggests that we should expect $\binom{2n}{k}/2^{2n}$ to be quite close to $\binom{2n}{n}/2^{2n}$ for k fairly close to n. If this is the case, then we might expect that the probability of getting some number of heads between $n - n_0$ and $n + n_0$ to be somewhat close to $2n_0$ times the probability of getting n heads, or somewhat close to $2n_0/\sqrt{n\pi}$. If this is true, then by the time n_0 gets up to somewhere around $\sqrt{n}$, the probability of getting some number of heads between $n - n_0$ and $n + n_0$ should be somewhat close to 1.

This intuition can be made precise, in a result called the *central limit theorem*, one of the most important results in probability. One very specific corollary of the central limit theorem is that if a coin is tossed $2n$ times, then for any constant C the probability of getting between $n - C\sqrt{n}$ and $n + C\sqrt{n}$ heads is at least $1 - e^{-C^2/3}$. For example, with $n = 1,000,000$ and $C = 5$, on tossing a coin $2,000,000$ times, the probability of getting between $995,000$ and $1,005,000$ heads is at least $1 - e^{-25/3} \approx .99976$.