Math 10860, Honors Calculus 2

Homework 1 solutions

Due in class Friday January 25

Instructions

Please present your answers neatly and clearly. Make use of space to increase the clarity of your presentation.

*I strongly encourage you to leave wide margins, leave at least an inch of space at the end of each answer, and write large! Remember that the grader is older than you (by a year or two) and may already be suffering from eyestrain!*

Justify your non-obvious assertions —

*the homework is as much about showing me that you are mastering the topics of the course, as it is about getting the right answers.*

Be careful with the logical flow of your proof-based answers. Make sure that each statement you write fits in to the proof in a clear way — either as something which follows from previous statements, or whose truth would be enough to establish the truth of the result you are being challenged to prove. Use connective phrases (like ”from this it follows that”, or “it is now enough to prove ..., which we now do”, etc), to highlight the flow of the proof.

*Consider submitting your answers, to at least some of the questions, in LaTeX. I’ll make the LaTeX source of the homework available to you to get you started.*

Reading for this homework

Class notes Sections 11.1, 11.2 and 11.3, and/or Spivak Chapter 13.

Assignment

1. Without doing any computations, evaluate the following integrals (you can be informal here; I’m not looking for a watertight \( \varepsilon-\delta \) justification, but rather an explanation that shows me that you know what is going on with the integral, and its interpretation as an area):

(a) \( \int_{-1}^{1} x^3 \sqrt{1-x^2} \, dx \)
Solution: We have
\[
\int_{-1}^{1} x^3\sqrt{1-x^2} \, dx = \int_{-1}^{0} x^3\sqrt{1-x^2} \, dx + \int_{0}^{1} x^3\sqrt{1-x^2} \, dx.
\]
Now \(x^3\sqrt{1-x^2}\) is non-negative on \([0, 1]\), so the second integral on the right above is the area under \(y = x^3\sqrt{1-x^2}\), above the x-axis, between \(x = 0\) and \(x = 1\); call this region \(R\), with area \(a(R)\).

Also, \(x^3\sqrt{1-x^2}\) is non-positive on \([0, 1]\), so the first integral on the right above is the negative of the area above \(y = x^3\sqrt{1-x^2}\), below the x-axis, between \(x = -1\) and \(x = 0\); call this this region \(S\), with area \(a(S)\), so that the value of the integral is \(a(R) - a(S)\).

The function being integrated is odd (\(f(-x) = -f(x)\)), and so, by reflection though the point \((0, 0)\), the region \(R\) is mapped to the region \(S\). That means that they have the same areas, and so \(a(R) - a(S) = 0\). This is the value of the integral.

(b) \(\int_{-1}^{1} (x^5 + 3)\sqrt{1-x^2} \, dx\).

Solution: We have, using basic properties of the integral,
\[
\int_{-1}^{1} (x^5 + 3)\sqrt{1-x^2} \, dx = \int_{-1}^{1} x^5\sqrt{1-x^2} \, dx + 3 \int_{-1}^{1} \sqrt{1-x^2} \, dx.
\]
By the same reasoning as in the first part of the question, the value of the first integral on the right-hand side above is 0. The value of the second term is three times the area under \(y = \sqrt{1-x^2}\) (non-negative on \([-1, 1]\)) above the x-axis between \(x = -1\) and \(x = 1\). This is three times the area of a semi-circle of radius 1, or \(3 \cdot (\pi/2) = (3/2)\pi\). This is the value of the integral.

2. Let \(f, g : [a, b] \to \mathbb{R}\) both be bounded, and let \(m, m^f\) and \(m^g\) be given by

- \(m = \inf\{f(x) + g(x) : x \in [a, b]\}\)
- \(m^f = \inf\{f(x) : x \in [a, b]\}\)
- \(m^g = \inf\{g(x) : x \in [a, b]\}\).

(a) Show that \(m^f + m^g \leq m\) (we used this fact in the proof that \(\int_{a}^{b} (f+g) = \int_{a}^{b} f + \int_{a}^{b} g\)).

Solution: We have that for all \(x \in [a, b]\), \(f(x) \geq m^f\) (if there was an \(x \in [a, b]\) with \(f(x) < m^f\), then \(m^f\) would not be a lower bound on \(\{f(x) : x \in [a, b]\}\)), and for the same reason \(g(x) \geq m^g\). Combining, we see that for all \(x \in [a, b]\), \(f(x) + g(x) \geq m^f + m^g\). This says that \(m^f + m^g\) is a lower bound for \(\{f(x) + g(x) : x \in [a, b]\}\). It follows that \(m\), the greatest lower bound, satisfies \(m \geq m^f + m^g\), as claimed.

(b) Show, by way of an example, that it is possible to have \(m^f + m^g < m\).

Solution: There are many examples that work. On \([0, 1]\), let \(f(x) = x\) and \(g(x) = 1 - x\), so that \((f + g)(x) = 1\). Then \(m^f = m^g = 0\) and \(m = 1\).
3. (a) Which functions have the property that every lower Darboux sum equals every upper Darboux sum?

Solution: For this to happen, we must have that on every interval the infimum of the function equals the supremum. This can only happen if the function is constant.

(b) Which functions have the property that there is some lower Darboux sum that equals some upper Darboux sum?

Solution: Suppose $L(f, P_1) = U(f, P_2)$ for some partitions $P_1$ and $P_2$. Consider the partition $P = P_1 \cup P_2$. We have

$$L(f, P_1) \leq L(f, P_1 \cup P_2) \leq U(f, P_1 \cup P_2) \leq U(f, P_2) = L(f, P_1),$$

so that

$$L(f, P_1 \cup P_2) = U(f, P_1 \cup P_2).$$

Thus there is at least one partition for which the lower Darboux sum equals the upper Darboux sum (for that single partition).

Let $P$ be such a partition, with $L(f, P) = U(f, P)$. Then on each subinterval $[t_{i-1}, t_i]$, $f$ is constant (since $\inf\{f(x) : x \in [t_{i-1}, t_i]\} = \sup\{f(x) : x \in [t_{i-1}, t_i]\}$). But the intervals $[t_{i-1}, t_i], i = 1, 2, \ldots, n$ overlap and cover $[a, b]$, so in fact $f$ is constant on $[a, b]$.

(c) Which continuous functions have the property that all lower Darboux sums are equal?

Solution: Only constant functions have this property. That all constant functions do is evident. In the other direction, suppose that $f$ is not constant on $[a, b]$. Let $m$ be the minimum value of $f$ on $[a, b]$ (minimum, not infimum: extreme value theorem!), and let $M$ be the maximum, achieved at $c$. Let $\varepsilon$ be smaller than $M - m$, but still positive. There is an interval that includes $c$ with the property that for $x$ in that interval, $f(x) > M - \varepsilon > m$ (this uses continuity of $f$), so the infimum of $f(x)$ for $x$ in that interval is greater than $m$. If $P$ is any partition of $[a, b]$ that has this interval as one of its intervals, then $L(f, P) > m(b - a)$. But if $Q$ is the partition $a, b$, then $L(f, Q) = m(b - a)$, showing that for non-constant continuous $f$, it cannot be the case that all lower Darboux sums are equal.

4. (a) Suppose $f$ is bounded and integrable on $[a, b]$, and that $m$ is a lower bound for $f$ on $[a, b]$ and $M$ an upper bound. Show that

$$m(b - a) \leq \int_a^b f \leq M(b - a).$$

Solution: If $m$ is a lower bound for $f$ on $[a, b]$, then it is also a lower bound for $f$ on any interval $[t_{i-1}, t_i]$ contained in $[a, b]$. So for any partition $P$ we have

$$m(t_i - t_{i-1}) \leq m_i(t_i - t_{i-1})$$

and so

$$m(b - a) = m \sum_{i=1}^n (t_i - t_{i-1}) = \sum_{i=1}^n m(t_i - t_{i-1}) \leq \sum_{i=1}^n m_i(t_i - t_{i-1}) \leq \int_a^b f.$$
A similar argument gives \( \int_a^b f \leq M(b - a) \).

(b) With the same hypotheses as for the last part, show that there exists a number \( \mu \), satisfying \( m \leq \mu \leq M \), such that

\[
\int_a^b f(x) \, dx = \mu(b - a).
\]

**Solution:** When \( a = b \) the result is trivial. For \( a < b \), this is immediate from the previous part, which asserts so that (as long as \( b > a \); when \( b = a \) the result is trivial)

\[
m \leq \frac{1}{b - a} \int_a^b f(x) \, dx \leq M.
\]

We take \( \mu = \left( \int_a^b f(x) \, dx \right) / (b - a) \), so that certainly \( \int_a^b f(x) \, dx = \mu(a - b) \), and the above inequality shows that \( m \leq \mu \leq M \).

(c) Show that if \( f \) is integrable on \([a,b]\), and if \( f(x) \geq 0 \) for all \( x \in [a,b] \), then \( \int_a^b f \geq 0 \).

**Solution:** If \( f(x) \geq 0 \) for all \( x \in [a,b] \), then for any subinterval of the form \([t_{i-1}, t_i]\) we have \( f(x) \geq 0 \) on the whole subinterval, so \( 0 \) is a lower bound for \( \{f(x) : x \in [t_{i-1}, t_i]\} \), so the greatest lower bound is non-negative. In the standard notation, then, the lower Darboux sum for any partition \( P \), \( L(f,P) = \sum_{i=1}^{n} m_i(t_i - t_{i-1}) \), has \( m_i \geq 0 \) for all \( i \), so \( L(f,P) \geq 0 \). Hence \( \int_a^b f \), being the supremum over all lower Darboux sums, is non-negative.

(d) Prove that if \( f \) and \( g \) are both integrable on \([a,b]\), and if \( f(x) \geq g(x) \) for all \( x \in [a,b] \), then \( \int_a^b f \geq \int_a^b g \).

**Solution:** If \( f \) and \( g \) are both integrable on \([a,b]\) then so is \( f - g \), and since \( f - g \geq 0 \) on \([a,b]\) we have from the first part that \( \int_a^b (f - g) \geq 0 \). But then from the basic properties of the integral it follows that \( \int_a^b f \geq \int_a^b g \).

5. Suppose that \( f \) is non-decreasing on \([a,b]\). The aim of this question is to show that \( f \) is integrable on \([a,b]\) (without any assumption on continuity of \( f \)).

(a) Prove that \( f \) is bounded on \([a,b]\).

**Solution:** Because \( f \) is non-decreasing on \([a,b]\), we have that for all \( x \in [a,b] \), \( f(a) \leq f(x) \leq f(b) \), so \( f(a) \) is a lower bound for \( f \) on \([a,b]\) and \( f(b) \) is an upper bound.

(b) If \( P = t_0, t_1, \ldots, t_n \) is a partition of \([a,b]\), what are \( L(f,P) \) and \( U(f,P) \)?

**Solution:** Because \( f \) is non-decreasing, on any interval \([t_{i-1}, t_i]\) we have that \( f(t_{i-1}) \) is a lower bound for \( \{ f(x) : x \in [t_{i-1}, t_i] \} \). It is the greatest lower bound,
for if $\alpha > f(t_{i-1})$ then $\alpha$ is not a lower bound for the set (being larger than $f(t_{i-1})$). Similarly, $f(t_i)$ is least upper bound for $\{f(x) : x \in [t_{i-1}, t_i]\}$. If follows that

$$L(f, P) = \sum_{i=1}^{n} f(t_{i-1})(t_i - t_{i-1})$$

and

$$U(f, P) = \sum_{i=1}^{n} f(t_i)(t_i - t_{i-1})$$

(c) Suppose that $P$ is such that $t_i - t_{i-1} = \delta$ for all $i$, for some $\delta > 0$. Calculate $U(f, P) - L(f, P)$ as a short, explicit expression, involving $\delta, a$ and $b$, that doesn’t involve a summation.

**Solution:** From the last part we have

$$L(f, P) = \sum_{i=1}^{n} f(t_{i-1})(t_i - t_{i-1}) = \sum_{i=1}^{n} f(t_{i-1})\delta = \delta \sum_{i=1}^{n} f(t_{i-1}),$$

and

$$U(f, P) = \delta \sum_{i=1}^{n} f(t_i).$$

Notice that, if the partition has $n$ parts (so $\delta = (b-a)/n$), then

$$L(f, P) = \delta(f(a) + f(t_1) + f(t_2) + \cdots + f(t_{n-1}))$$

while

$$L(f, P) = \delta(f(t_1) + f(t_2) + \cdots + f(t_{n-1}) + f(b)),$$

so that

$$U(f, P) - L(f, P) = \delta(f(b) - f(a)).$$

(d) Prove that $f$ is integrable on $[a, b]$.

**Solution:** Given $\varepsilon > 0$ there is $n \in \mathbb{N}$ large enough that $(b-a)(f(b) - f(a))/n < \varepsilon$. Consider the partition $P$ that divides $[a, b]$ into $n$ equal subintervals, each of length $(b-a)/n$. By the last part, for this partition we have

$$U(f, P) - L(f, P) = (b-a)(f(b) - f(a))/n < \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, we conclude that $f$ is integrable on $[a, b]$.

(e) Give an example of a bounded non-decreasing function on $[0, 1]$ which is discontinuous at infinitely many points (such a function is still integrable, by the last part of the question).
Solution: There are many, many such examples. Here’s one:

\[
f(x) = \begin{cases} 
0 & \text{if } 0 \leq x < 1/2 \\
1/2 & \text{if } 1/2 \leq x < 2/3 \\
2/3 & \text{if } 2/3 \leq x < 3/4 \\
3/4 & \text{if } 3/4 \leq x < 4/5 \\
\vdots \\
(k - 1)/k & \text{if } (k - 1)/k \leq x < k/(k + 1), \ k = 1, 2, 3, \ldots \\
1 & \text{if } x = 1.
\end{cases}
\]

This is non-decreasing and has discontinuities at 1/2, 2/3, 3/4, 4/5, \ldots.

6. Define \( f : [0, 1] \to \mathbb{R} \) by

\[
f(x) = \begin{cases} 
x & \text{if } x \text{ is rational} \\
0 & \text{if } x \text{ is irrational}
\end{cases}
\]

(a) Compute \( L(f, P) \) for every partition \( P \) of \([0, 1]\).

Solution: The irrational numbers are dense in the reals, so there is one in every interval in any partition \( P \). This means that \( L(f, P) = 0 \) for every partition \( P \).

(b) Find \( \inf \{ U(f, P) : P \text{ a partition of } [0, 1] \} \).

Solution: We claim that \( \inf \{ U(f, P) : P \text{ a partition of } [0, 1] \} = 1/2 \). One way to see this is to observe that for any interval \([a, b]\),

\[
\sup \{ f(x) : x \in [a, b] \} = \sup \{ x : x \in [a, b] \} = b.
\]

That \( \sup \{ x : x \in [a, b] \} = b \) is obvious. That \( b \) is an upper bound for \( \{ f(x) : x \in [a, b] \} \) is also obvious. If \( b' < b \) then there is a rational \( b'' < b' < b'' < b \) and also with \( a < b' < b \) (by density of rationals in the reals); since \( f(b'') = b'' > b' \), it can’t be that \( b' \) is also an upper bound for \( \{ f(x) : x \in [a, b] \} \), from which it follows, as claimed, that \( \sup \{ f(x) : x \in [a, b] \} = b \).

Now using that \( \sup \{ f(x) : x \in [a, b] \} = \sup \{ x : x \in [a, b] \} \) for any interval, we conclude that for any partition \( P \) of \([0, 1]\),

\[
U(f, P) = U(id, P)
\]

where \( id : [0, 1] \to \mathbb{R} \) is the identity function \( id(x) = x \). We know that

\[
\inf \{ U(id, P) : P \text{ a partition of } [0, 1] \} = \int_0^1 x \, dx = \frac{1}{2},
\]

so indeed

\[
\inf \{ f, P : P \text{ a partition of } [0, 1] \} = \frac{1}{2}.
\]

(c) Does \( \int_0^1 f \) exist, and if so what is its value?

Solution: The integral does not exist since the supremum of lower Darboux sums is 0, different from the infimum of upper Darboux sums (1/2).
An extra credit problem:

A Prove that for all $a \geq 1$, $\int_1^a \frac{dt}{t}$ exists. (It goes without saying, but I’ll say it anyway: I’m expecting you to use the basic properties of the integral here, not any witchy properties of and weird and exotic functions that you might learned about in a previous life!)

B For $a, b \geq 1$, prove that

$$\int_1^a \frac{dt}{t} + \int_1^b \frac{dt}{t} = \int_1^{ab} \frac{dt}{t}.$$