Math 10860, Honors Calculus 2

Homework 4  

Due in class Friday February 15

Instructions

New note: remember the comment I made by email this week, based on feedback from the graders:

It seems there is a growing tendency to use the “three dots” symbol, to avoid writing too much. This isn’t consistent with how mathematics is presented in modern times (go to the library, grab a random journal from a shelf, and skim as many pages as you like — you won’t find this symbol). In particular, please try to curtail the use of that symbol; in general, remember the convention that properly presented mathematical arguments should read like prose. Once you have finished writing a proof, you should read it out loud. It should make sense as an english-language narrative!

Please present your answers neatly and clearly. Make use of space to increase the clarity of your presentation.

I strongly encourage you to leave wide margins, leave at least an inch of space at the end of each answer, and write large! Remember that the grader is older than you (by a year or two) and may already be suffering from eyestrain!

Justify your non-obvious assertions —

the homework is as much about showing me that you are mastering the topics of the course, as it is about getting the right answers.

Be careful with the logical flow of your proof-based answers. Make sure that each statement you write fits in to the proof in a clear way — either as something which follows from previous statements, or whose truth would be enough to establish the truth of the result you are being challenged to prove. Use connective phrases (like ”from this it follows that”, or “it is now enough to prove ..., which we now do”, etc), to highlight the flow of the proof.

Consider submitting your answers, to at least some of the questions, in LaTeX. I'll make the LaTeX source of the homework available to you to get you started.

Reading for this homework

Class notes Sections 11.7 and Chapter 12 (my Chapter 12 is covered also by Spivak’s Chapter 12).
Assignment

1. Decide whether or not the following improper integrals exist.
   (a) \( \int_0^\infty \frac{dx}{\sqrt{1+x^3}} \).
   (b) \( \int_0^\infty \frac{dx}{x\sqrt{1+x}} \).

2. Suppose \( \int_{-\infty}^{\infty} f \) exists. Let \( h, g \) be functions with \( h(N) \to -\infty \) and \( g(N) \to +\infty \) as \( N \to +\infty \). Prove that
   \[ \lim_{N \to \infty} \int_{h(N)}^{g(N)} f \]
   exists and equals \( \int_{-\infty}^{\infty} f \).

3. Find \( f^{-1} \) for each of the following \( f \). Specify the domain and range of \( f^{-1} \) in each case.
   (a) \( f(x) = x^3 + 1 \).
   (b) \( f(x) = \begin{cases} x & \text{if } x \text{ is rational} \\ -x & \text{if } x \text{ is irrational} \end{cases} \)
   (c) \( f(x) = x + [x] \). (Remember that \([x]\) is the largest integer less than or equal to \( x \).)
   (d) \( f(x) = \frac{x}{1-x^2}, -1 < x < 1 \).

4. Suppose \( f \) and \( g \) are increasing.
   (a) Is \( f + g \) necessarily increasing?
   (b) Is \( fg \) necessarily increasing?
   (c) Is \( f \circ g \) necessarily increasing?

5. On which intervals \([a, b]\) will the following functions by one-to-one?
   (a) \( f(x) = x^3 - 3x^2 \).
   (b) \( f(x) = (1 + x^2)^{-1} \).

6. Find a formula for \((f^{-1})''(x)\), and decide under what circumstances the derivative actually exists.

7. Suppose that \( f : [a, b] \to [c, d] \) is increasing, and integrable on \([a, b] \). Prove that \( f^{-1} : [c, d] \to [a, b] \) is integrable on \([c, d] \), and that in fact
   \[ \int_a^b f + \int_c^d f^{-1} = bd - ac. \]

   **Big hint:** draw a picture, and include a partition \( P \) of \([a, b]\) in the picture.

   **An extra credit problem:** We have seen that integrability on a closed interval \([a, b]\) is closed under
• linear combinations
• multiplication
• changing values at finitely many places.

Is integrability closed under taking compositions? Specifically, suppose that $f : [0, 1] \to [0, 1]$ and $g : [0, 1] \to [0, 1]$ are both integrable. Is $f \circ g$ necessarily integrable on $[0, 1]$?