Instructions
Same as always.

Reading for this homework
Spivak, Chapter 19 (up to and including integration by parts), and/or the class notes Section 14.1 through 14.3.

Assignment
1. Here are a few “standard” integration formulae, randomly culled from the back page of a calculus textbook. Verify that each of them is correct. Part of this entails checking that both the function being integrated and the proposed antiderivative have the same domain (part of the full answer will be a statement of that domain); the other part entails checking that at each point in the domain of the proposed antiderivative, the derivative of the proposed antiderivative is the function being integrated. These should be very easy, but a little care might be required for the antiderivatives that involve absolute values.

(a) \[ \int \cot x \, dx = \log |\sin x|. \]

Solution: \( \cot \) has domain \( \mathbb{R} \setminus \{ n\pi : n \in \mathbb{Z} \} \) (we have to rule out the places where \( \sin \) (in the denominator of \( \cot \)) is zero). \( \log |\sin x| \) also has domain \( \mathbb{R} \setminus \{ n\pi : n \in \mathbb{Z} \} \) (we have to rule out the places where \( \sin \) (in the log) is zero; at all other inputs \( \log |\sin| \) makes sense).

For all \( x \) for which \( \sin \) is positive, \( \log |\sin x| = \log(\sin x) \), and the derivative of this function is \( (1/\sin x) \cos x = \cot x \). For all \( x \) for which \( \sin \) is negative, \( \log |\sin x| = \log - (\sin x) \), and the derivative of this function is \( (1/ -\sin x)(-\cos x) = \cot x \). So for all \( x \) (for which \( \sin \) is not zero), the derivative of \( \log |\sin x| \) is \( \cot \).

(b) \[ \int \sec x \, dx = \log |\sec x + \tan x|. \]

Solution: \( \sec \) has domain \( \mathbb{R} \setminus \{(n + 1/2)\pi : n \in \mathbb{Z} \} \) (we have to rule out the places where \( \cos \) (in the denominator of \( \sec \)) is zero). Clearly \( \log |\sec x + \tan x| \) is...
also not defined at the set of points \( \{(n+1/2)\pi : n \in \mathbb{Z}\} \) — these are exactly the points where neither \( \sec x \) nor \( \tan x \) are defined.

It remains to check that at all other points, \( \log |\sec x + \tan x| \) is defined. This requires checking that \( \sec x + \tan x \neq 0 \) at all \( x \) for which \( \cos x \neq 0 \). But

\[
\sec x + \tan x = \frac{1 + \sin x}{\cos x},
\]

and this is only zero when \( \sin x = -1 \). At all \( x \) for which \( \sin x = -1 \), we have \( \cos x = 0 \), so we indeed conclude that at all points with \( \cos x \neq 0 \), we have \( \sec x + \tan x \neq 0 \). This concludes the demonstration that \( \sec \) and \( \log |\sec + \tan| \) have the same domains.

For all \( x \) for which \( \sec x + \tan x \) is defined and positive, \( \log |\sec x + \tan x| = \log(\sec x + \tan x) \). To differentiate this, it’s easiest to write it as \( \log((1+\sin x) / \cos x) \).

The derivative is

\[
\left( \frac{\cos x}{1 + \sin x} \right) \left( \frac{(\cos x)(\cos x) + (1 + \sin x)(\sin x)}{\cos^2 x} \right) = \left( \frac{1}{1 + \sin x} \right) \left( \frac{\cos^2 x + \sin^2 x + \sin x}{\cos x} \right)
\]

\[
= \frac{1}{\sec x}.
\]

A similar calculation goes through for those \( x \) for which \( \sec x + \tan x \) is defined and negative.

(c) \( \int \sin^{-1} x \ dx = x \sin^{-1} x + \sqrt{1-x^2} \).

**Solution:** The domain of both sides is clearly \([−1, 1]\). Recalling that the derivative of \( \sin^{-1} x \) is \( 1/\sqrt{1-x^2} \), we get that the derivative of \( x \sin^{-1} x + \sqrt{1-x^2} \) is

\[
\frac{x}{\sqrt{1-x^2}} + \sin^{-1} x - \frac{x}{\sqrt{1-x^2}} = \sin^{-1} x.
\]

(d) For an arbitrary real \( a \) (positive or negative), \( \int \frac{dx}{x^2-a^2} = \frac{1}{2a} \log \left| \frac{x-a}{x+a} \right| \).

**Solution:** The domain of the integrand is all reals except \(-a\) and \(a\). These are exactly the places where we cannot evaluate \( \log |(x-a)/(x+a)| \): at \( x = a \neq 0 \) we are looking at \( \log 0 \) (which doesn’t exist) and at \( x = -a \neq 0 \) even the expression \( (x-a)/(x+a) \) doesn’t make sense. At all other \( x \), \( |(x-a)/(x+a)| > 0 \) so \( \log |(x-a)/(x+a)| \) is defined.

(Note that we don’t discuss \( a = 0 \): the equality makes no sense in this case, since the right-hand side is undefined. I think that ignoring \( a = 0 \) was implicit in the parenthetical comment “(positive or negative)” in the question).

Fix \( a \neq 0 \). For those \( x \neq a, -a \) for which \( (x-a)/(x+a) > 0 \), we have that \( (1/2a) \log |(x-a)/(x+a)| = (1/2a) \log(x-a)/(x+a) \), and the derivative of this is

\[
(1/2a) \left( \frac{x+a}{x-a} \right) \left( \frac{(x+a)(x-a)}{(x-a)^2} \right) = (1/2a) \left( \frac{(x+a)^2}{(x-a)(x+a)^2} \right)
\]

\[
= \frac{1}{x+a(x-a)}
\]

\[
= \frac{1}{x^2-a^2}.
\]
For those \( x \neq a, -a \) for which \( (x - a)/(x + a) < 0 \), we have that \((1/2a) \log |(x - a)/(x + a)| = (1/2a) \log -(x - a)/(x + a)\), and the derivative calculation goes through the same way, except that we multiply by two minus signs. We end up with \(1/(x^2 - a^2)\) again.

2. (a) In class we found \( \int \frac{\log x}{x} \, dx \). Now find \( \int \frac{1}{x \log x} \, dx \). (A little care might be needed.)

Solution: I don’t see how to do this with integration by parts. An obvious substitution is \( u = \log x \) (which is defined, continuous, invertible, differentiable, derivative never zero on \((0, \infty)\), exactly the domain of the integrand, hence the substitution is valid). After this substitution the integral becomes

\[
\int \frac{du}{u} = \log |u|
\]

(Note that we need the \(|u|\) here: as \( x \) ranges over \((0, \infty)\), \( \log x \) takes on both positive and negative values; this was where the slight care was required). Re-expressing in terms of \( x \), we get

\[
\int \frac{1}{x \log x} \, dx = \log |\log x|
\]

(not \( \log(\log x) \)). Note that the domain of both sides is \((0, 1) \cup (1, \infty)\).

(b) We’ve seen that \( \int_1^\infty \frac{dx}{x} \) goes off to infinity, but not \( \int_1^\infty \frac{dx}{x^{1+\varepsilon}} \), for any \( \varepsilon > 0 \). In other words, increasing the denominator from \( x \) to \( x^{1+\varepsilon} \) causes the area trapped under the curve (between \( e \) and \( \infty \)) to go from being infinite to being finite. What about increasing the denominator from \( x \) to \( x \log x \)? I.e., does \( \int_1^\infty \frac{1}{x \log x} \, dx \) exist?

Solution: We have, for any \( N \geq e \),

\[
\int_e^N \frac{1}{x \log x} \, dx = \log |\log N| - \log |\log e| = \log \log N.
\]

This goes to infinity (we can make it exceed 100, for example, by taking \( N > e^{100} \)), so

\[
\int_e^N \frac{1}{x \log x} \, dx
\]

does not exist.

(c) What about \( \int_1^e \frac{1}{x \log x} \, dx \)?

Solution: We have, for any \( e \geq \varepsilon > 1 \),

\[
\int_\varepsilon^e \frac{1}{x \log x} \, dx = \log |\log e| - \log |\log \varepsilon| = - \log \log \varepsilon.
\]

This goes to infinity as \( \varepsilon \) approaches 1. Indeed, as \( \varepsilon \) approaches 1 from above, \( \log \varepsilon \) approaches 0 from above, and so \( \log \log \varepsilon \) approaches \(-\infty\), and so \( - \log \log \varepsilon \) approaches \( \infty \). We conclude that

\[
\int_1^e \frac{1}{x \log x} \, dx
\]

does not exist.
3. This problem concerns a very important non-elementary function, called the *gamma* function.

(a) Show that for all $x > 0$, the integral

$$
\int_0^\infty e^{-t^{x-1}} dt
$$

is finite. The value of this integral, for each such $x$, is denoted $\Gamma(x)$.

**Solution:** We start by considering the case $x \geq 1$. Here the function $t \mapsto e^{-t^{x-1}}$ is defined, non-negative and continuous (so integrable) on $[0, N]$ for all $N \geq 0$. We’ll show that the integral $\int_0^\infty e^{-t^{x-1}} dt$ exists by finding a function $g : [0, \infty) \to \mathbb{R}$ such that

- $0 \leq e^{-t^{x-1}} \leq g(t)$ for all $t \geq 0$ and
- $\int_0^\infty g(t) \, dt$ exists (is finite).

By a comparison lemma that we proved in class, this is enough to show that $\int_0^\infty e^{-t^{x-1}} dt$ exists.

Observe that $e^{-t^{x-1}}$ is a ratio:

$$\frac{t^{x-1}}{e^t}.$$

We instead (temporarily) consider the ratio

$$\frac{t^{x-1}}{e^{t/2}}.$$

Since $x \geq 1$, the power in the polynomial in the denominator is non-negative. By modifying an earlier proof (showing that $\lim_{t \to \infty} t^n / e^n = 0$ for $n \geq 1$, $n \in \mathbb{N}$), we can show that $t^{x-1}/e^{t/2} \to 0$ as $t \to \infty$. In particular, there is $t_0 \geq 0$ such that

$$\frac{t^{x-1}}{e^{t/2}} \leq 1$$

for all $t \geq t_0$, so

$$\frac{t^{x-1}}{e^t} \leq e^{-t/2}.$$

Now consider the function $g : [0, \infty) \to \mathbb{R}$ defined by

$$g(x) = \begin{cases} 
   e^{-t^{x-1}} & \text{if } t < t_0 \\
   e^{-t/2} & \text{if } t \geq t_0.
\end{cases}$$

By construction, $0 \leq e^{-t^{x-1}} \leq g(t)$ for all $t \geq 0$. For $N > t_0$,

$$\int_0^N g(t) \, dt = \int_0^{t_0} g(t) \, dt + \int_{t_0}^N e^{-t/2} \, dt = C + \left[-2e^{-t/2}\right]_{t=t_0}^N = C' + 2/e^{N/2}$$

where $C, C'$ are some constants. Since $2/e^{N/2} \to 0$ as $N \to \infty$ we get that $\int_0^\infty g(t) \, dt$ exists, and this completes the proof that $\int_0^\infty e^{-t^{x-1}} dt$ exists.
Now we consider $0 < x < 1$. Here we have to consider $\int_0^1$ and $\int_1^\infty$ separately, since for $0 < x < 1$, the function $t \mapsto e^{-t}x^{-1}$ goes to infinite as $t$ goes to 0 (from above). In this regime, for $t \geq 1$ we have

$$e^{-t}x^{-1} \leq e^{-t},$$

so using an argument similar to the case $x \geq 1$ we get that $\int_1^\infty e^{-t}x^{-1}$ exists. It’s left to consider $\int_0^1 e^{-t}x^{-1} dt$. On $[0, 1]$, $e^{-t} \leq 1$, so

$$0 \leq e^{-t}x^{-1} \leq t^{-1}.$$ 

So if we can show that $\int_0^1 t^{-x} dt$ exists, by comparison so does $\int_0^1 e^{-t}x^{-1} dt$.

We have, for $0 < \varepsilon < 1$,

$$\int_\varepsilon^1 t^{-x} dt = \left[\frac{t^x}{x}\right]_{t=\varepsilon}^1 = \frac{1}{x} - \frac{\varepsilon^x}{x} \to \frac{1}{x}$$

as $\varepsilon \to 0$, so $\int_0^1 t^{-x} dt$, and hence $\int_0^\infty e^{-t}x^{-1} dt$, exist.

(b) Prove that for all $x > 0$, $\Gamma(x + 1) = x\Gamma(x)$.

**Solution:** Fix $x > 0$, so $x + 1 > 0$ also. We consider, for $N > 0$, the integral

$$\int_0^N e^{-t}x dt.$$ 

We evaluate this using integration by parts, taking $f(t) = t^x$ (so $f'(t) = xt^{x-1}$) and $g'(t) = e^{-t}$ (so $g(t) = -e^{-t}$). We get

$$\int_0^N e^{-t}x dt = -\left[e^{-t}x\right]_0^N + x \int_0^N e^{-t}x^{-1} dx = \frac{N^x}{eN} + x \int_0^N e^{-t}x^{-1} dx$$

As $N \to \infty$ the left-hand side approaches $\Gamma(x + 1)$ (by definition of the $\Gamma$ function). The integral on the right-hand side approaches $\Gamma(x)$ (again by definition). Because exponentials beat polynomials, the other term on the right-hand side approaches 0. So in the limit as $N \to \infty$ we get $\Gamma(x + 1) = x\Gamma(x)$, as required.

(c) Prove that $\Gamma(n) = (n - 1)!$ for all natural numbers $n$. (So, the gamma function is a continuous function that extends the factorial function to (almost) all reals).

**Solution:** We start by evaluating

$$\Gamma(1) = \int_0^\infty e^{-t} dt = -e^{-t}\big|_0^\infty = 0 - (-1) = 1 = 0!.$$ 

To get $\Gamma(n) = (n - 1)!$ for $n > 1$ is now a simple induction based on $\Gamma(x + 1) = x\Gamma(x)$.