Instructions

The usual, plus: a very large part of this homework is a list of integrals to compute (in this respect, it is the unique homework that you will see in an Honors class, that is similar to one you might see in regular calculus class). You could in principle do each of these by entering the integrand into Mathematica, noting the result, and then verifying it by differentiation. This is not what I’m intending. I want to see you tackle these integrals using integration by parts, integration by appropriate substitutions, and/or integration by partial fractions. For each integral, you should say clearly what method/substitution you are using in each step; other than that, no great explanation is need.

Nota bene: I promise that all the integrals below, except the two pairs I specifically highlight, have elementary primitives. I don’t promise that the homework is typo-free, and unfortunately even a tiny typo can turn a do-able integration into an impossible one; so alert me if you think that there is a problem with any of these!

Reading for this homework

Section 14 of the course notes, and/or Chapter 19 of Spivak.

Assignment

Appetizer (Not to be handed in!): As a warm-up for the long list of integrals, you could do Spivak, Chapter 19, Questions 1 and 2, 3rd ed. (twenty integrals involving simple algebraic manipulation and quick substitutions).

Main course: For each of the 8 questions below (all three-parters), turn in any two parts.

1. First, some problems that are best suited to integration by parts:

   (a) \[ \int x^2 \sin x \, dx. \]

   (b) \[ \int x (\log x)^2 \, dx. \]
(c) \[ \int \sec^3 x \, dx. \]

Here I strongly recommend using integration by parts, and not the substitution \( t = \tan(x/2) \), which leads to needing to solve a 6 by 6 system of linear equalities. This one is quite tricky. It might be helpful to use as a black box

\[ \int \sec x \, dx = \log |\sec x + \tan x|, \]

which can easily be checked by differentiation (I think that this was on an earlier homework).

2. Next, three integral identities/reduction formulae (none of \( \log(\log x) \), \( 1/(\log x) \), \( x^2e^{-x^2} \) nor \( e^{-x^2} \) have elementary primitives)

(a) Express \( \int \log(\log x) \, dx \) in terms of \( \int dx / \log x \).
(b) Express \( \int x^2e^{-x^2} \, dx \) in terms of \( \int e^{-x^2} \, dx \).
(c) Find a reduction formula for \( \int (\log x)^n \, dx \), and use it to calculate \( \int (\log x)^3 \, dx \).

3. Next, some problems involving substitutions such as \( x = \sin u \), \( x = \cos u \): (As well as knowing \( \int \sec dx \), it might be helpful here to know

\[ \int \csc x \, dx = -\log |\csc x + \cot x|, \]

which can also be verified easily by differentiation.)

(a) \[ \int \frac{dx}{\sqrt{1 - x^2}}. \]
(b) \[ \int \frac{dx}{x\sqrt{x^2 - 1}}. \]
(c) \[ \int x^3\sqrt{1 - x^2} \, dx. \]

This will also involve the integration of powers of sin and cos.

4. Next, a collection of integrals calling for a variety of substitutions: (Remember that there are no silver-bullet rules for substitution. Just try to substitute for an expression that appears frequently or prominently. If two different troublesome expressions appear, try to express them both in terms of some new expression.)
(a) \[ \int \frac{dx}{\sqrt{1 + e^x}}. \]

(b) \[ \int \frac{4^x + 1}{2^x + 1} \, dx. \]

(c) \[ \int \frac{1}{x^2} \sqrt{\frac{x-1}{x+1}} \, dx. \]

5. Next, some integrals where it might not be too ridiculous to consider the “magic bullet” substitution \( t = \tan(x/2) \).

(a) \[ \int \frac{dx}{a \sin x + b \cos x}. \quad (a, b \text{ arbitrary constants}) \]

(b) (A slight variant of the “magic bullet” substitution might be in order here):
\[ \int \frac{dx}{1 - \sin^2 x}. \]

(c) \[ \int \frac{dx}{3 + 5 \sin x}. \]

6. Next, some integrands appropriate for partial fractions:

(a) \[ \int \frac{2x^2 + 7x - 1}{x^3 - 3x^2 + 3x - 1} \, dx. \]

(b) \[ \int \frac{3x^2 + 3x + 1}{x^3 + 2x^2 + 2x + 1} \, dx. \]

(c) \[ \int \frac{3x}{(x^2 + x + 1)^3} \, dx. \]

7. Next, a pot-pourri with a (slightly non-obvious) trigonometric flavor.

(a) \[ \int \sqrt{1 - 4x - 2x^2} \, dx. \]

(b) \[ \int \cos x \sqrt{9 + 25 \sin^2 x} \, dx. \]
8. Finally, another pot-pourri. Who knows what methods might be needed?

(a) 
\[
\int \frac{x \arctan x}{(1 + x^2)^3} \, dx.
\]

(b) 
\[
\int \log \sqrt{1 + x^2} \, dx.
\]

(c) 
\[
\int \sqrt{\tan x} \, dx.
\]

Dessert (Not to be handed in!): If all this wasn’t enough, you could also look at Spivak, Chapter 19, Questions 8 and 9, 3rd ed.