1. First, some problems that are best suited to integration by parts:

(a) \[ \int x^2 \sin x \, dx. \]

**Solution:** Via integration by parts, first with \( f(x) = x^2 \) and \( g'(x) = \sin x \), and then with \( f(x) = x \) and \( g'(x) = \cos x \), have

\[
\int x^2 \sin x \, dx = -x^2 \cos x + 2 \int x \cos x \, dx
\]

\[
= -x^2 \cos x + 2 \left( x \sin x - \int \sin x \, dx \right)
\]

\[
= -x^2 \cos x + 2x \sin x + 2 \cos x.
\]

(b) \[ \int x (\log x)^2 \, dx. \]

**Solution:** Via integration by parts, with \( f(x) = (\log x)^2 \) and \( g'(x) = x \), have

\[
\int x (\log x)^2 \, dx = \frac{x^2}{2} (\log x)^2 \left( \int \frac{2 \log x \, x^2}{2} \, dx \right)
\]

\[
= \frac{x^2}{2} \log^2 x - \int x \log x \, dx.
\]

To evaluate \( \int x \log x \, dx \), use integration by parts, with \( f(x) = \log x \) and \( g'(x) = x \), so

\[
\int x \log x \, dx = (\log x) \frac{x^2}{2} - \int \frac{1}{x} \frac{x^2}{2} \, dx = \frac{x^2 \log x}{2} - \frac{x^2}{4}.
\]

So

\[
\int x (\log x)^2 \, dx = \frac{x^2 \log^2 x}{2} - \frac{x^2 \log x}{2} + \frac{x^2}{4}.
\]
\[ \int \sec^3 x \, dx. \]

Here I strongly recommend using integration by parts, and not the substitution \( t = \tan(x/2) \), which leads to needing to solve a 6 by 6 system of linear equalities. This one is quite tricky. It might be helpful to use as a black box

\[ \int \sec x \, dx = \log |\sec x + \tan x|, \]

which can easily be checked by differentiation (I think that this was on an earlier homework).

**Solution:** Via integration by parts, with \( u = \sec x \) and \( dv = \sec^2 x \, dx \) (so \( du = \sec x \tan x \, dx \) and \( v = \tan x \)), have

\[
\int \sec^3 x \, dx = \sec x \tan x - \int \sec x \tan^2 x \, dx \\
= \sec x \tan x - \int \sec x (\sec^2 - 1) \, dx \\
= \sec x \tan x - \int \sec^3 x \, dx + \int \sec x \, dx \\
= \sec x \tan x + \log |\sec x + \tan x| - \int \sec^3 x \, dx,
\]

so

\[
\int \sec^3 x \, dx = \frac{1}{2} \left( \sec x \tan x + \log |\sec x + \tan x| \right).
\]

2. Next, three integral identities/reduction formulae (none of \( \log(\log x) \), \( \frac{1}{\log x} \), \( x^2 e^{-x^2} \) nor \( e^{-x^2} \) have elementary primitives).

(a) Express \( \int \log(\log x) \, dx \) in terms of \( \int dx/\log x \).

**Solution:** Use integration by parts with \( dv = dx \) (so \( v = x \)) and \( u = \log(\log x) \) (so \( du = 1/(x \log x) \)), to get

\[
\int \log(\log x) \, dx = x \log(\log x) - \int \frac{dx}{\log x}.
\]

(b) Express \( \int x^2 e^{-x^2} \, dx \) in terms of \( \int e^{-x^2} \, dx \).

**Solution:** Use integration by parts with \( u = x \) (so \( du = dx \)) and \( dv = x e^{-x^2} \, dx \) (so \( v = -e^{-x^2}/2 \)) to get

\[
\int x^2 e^{-x^2} \, dx = -\frac{xe^{-x^2}}{2} + \frac{1}{2} \int e^{-x^2} \, dx.
\]

(c) Find a reduction formula for \( \int (\log x)^n \, dx \), and use it to calculate \( \int (\log x)^3 \, dx \).

**Solution:** Let \( L_n = \int (\log x)^n \, dx \). We have
• \( L_0 = \int 1 \, dx = x, \)
• \( L_1 = \int \log x \, dx = \int (1 \cdot \log x) \, dx = x \log x - x \) (integration by parts with \( u = \log x \, dv = dx \)), and,
• for \( n \geq 2 \), using integration by parts with \( u = \log^{n-1} x \) (so \( du = \frac{(n-1)(\log^{n-2} x)}{x} \)), \( dv = \log x \, dx \) (so \( v = x(\log x) - x \), as we just saw), get:

\[
L_n = \int (\log x)^n \, dx
\]

\[
= x \log^n x - x \log^{n-1} x - (n-1) \int (\log^{n-1} x - \log^{n-2} x) \, dx
\]

\[
= x \log^n x - x \log^{n-1} x - (n-1) (L_{n-1} - L_{n-2}).
\]

This gives

\[
L_2 = x \log^2 x - x \log x - (x(\log x) - x) = x \log^2 x - 2x \log x + 2x,
\]

and

\[
L_3 = x \log^3 x - x \log^2 x - 2(x \log^2 x - 2x \log x + 2x - x \log x + x)
\]

\[
= x \log^3 x - 3x \log^2 x + 6x \log x - 6x
\]

There is more than one way to skin this cat; here is a simpler way. We have

• \( L_0 = \int 1 \, dx = x, \) and,
• for \( n \geq 1 \), using integration by parts with \( u = \log^n x \) (so \( du = n(\log^{n-1} x)/x \)), \( dv = dx \) (so \( v = x \)), get:

\[
L_n = \int (\log x)^n \, dx
\]

\[
= x \log^n x - n \int \log^{n-1} x \, dx
\]

\[
= x \log^n x - nL_{n-1}.
\]

This gives

\[
L_1 = x \log x - x,
\]

\[
L_2 = x \log^2 x - 2x \log x + 2x,
\]

and

\[
L_3 = x \log^3 x - 3x \log^2 x + 6x \log x - 6x.
\]

3. Next, some problems involving substitutions such as \( x = \sin u, \ x = \cos u \): (As well as knowing \( \int \sec dx \), it \textit{might} be helpful here to know

\[
\int \csc x \, dx = -\log |\csc x + \cot x|,
\]

which can also be verified easily by differentiation.)
(a) \[ \int \frac{dx}{\sqrt{1 - x^2}}. \]

**Solution:** Via substitution \( x = \sin u \) (so \( dx = \cos u \, du \) and \( \sqrt{1 - x^2} = \cos x \)), get
\[
\int \frac{dx}{\sqrt{1 - x^2}} = \int \frac{\cos u \, du}{\cos u} = \int 1 \, du = u = \sin^{-1} x.
\]

(b) \[ \int \frac{dx}{x\sqrt{x^2 - 1}}. \]

**Solution:** Via substitution \( x = \sec u \) (so \( dx = \sec u \tan u \, du \) and \( \sqrt{x^2 - 1} = |\tan x| \)), get
\[
\int \frac{dx}{x\sqrt{x^2 - 1}} = \int \frac{\sec u \tan u \, du}{\sec u |\tan u|}.
\]
If \( x > 1 \) then \( \tan u > 0 \) and this becomes \( \int 1 \, du = u = \sec^{-1} x. \)
If \( x < -1 \) then \( \tan u < 0 \) and this becomes \( \int -1 \, du = -u = -\sec^{-1} x. \)
So:
\[
\int \frac{dx}{x\sqrt{x^2 - 1}} = \left\{ \begin{array}{ll} 
\sec^{-1} x & \text{if } x > 1 \\
-\sec^{-1} x & \text{if } x < -1.
\end{array} \right.
\]
(See comment on this in email from March 18.)

(c) \[ \int x^3 \sqrt{1 - x^2} \, dx. \]

This will also involve the integration of powers of \( \sin \) and \( \cos \).

**Solution:** Let \( x = \sin u \), so \( dx = \cos u \, du \) and \( x^3 \sqrt{1 - x^2} = \sin^3 u \cos u \). We get
\[
\int x^3 \sqrt{1 - x^2} \, dx = \int \sin^3 u \cos^2 u \, du
= \int \sin^2 u \cos^2 u \sin u \, du
= \int (1 - \cos^2 u) \cos^2 u \sin u \, du
= \int (\cos^2 u - \cos^4 u) \sin u \, du.
\]
Now make the substitution \( w = \cos u \), so \( -dw = \sin u \, du \), to get
\[
\int (\cos^2 u - \cos^4 u) \sin u \, du = -\int (w^2 - w^4) \, dw = \frac{w^5}{5} - \frac{w^3}{3} = \frac{\cos^5 u}{5} - \frac{\cos^3 u}{3}.
\]
We get
\[
\int x^3 \sqrt{1 - x^2} \, dx = \frac{\cos^5 (\arcsin x)}{5} - \frac{\cos^3 (\arcsin x)}{3}.
\]
This can be simplified: The domain of $x^3\sqrt{1-x^2}$ is $-1 \leq x \leq 1$. If $0 \leq x \leq 1$, then the angle $u$ lies between $0$ and $\pi/2$. Forming a right-angled triangle, with one angle $u$, opposite side $x$, hypotenuse $1$, adjacent side $\sqrt{1-x^2}$, we find that $\cos u = \sqrt{1-x^2}$. This says that for $0 \leq x \leq 1$,

$$
\int x^3\sqrt{1-x^2} \, dx = \frac{(\sqrt{1-x^2})^5}{5} - \frac{(\sqrt{1-x^2})^3}{3}.
$$

Differentiating this expression on the whole interval $[-1, 1]$ shows that the above expression is valid for all $x \in [-1, 1]$.

4. Next, a collection of integrals calling for a variety of substitutions: (Remember that there are no silver-bullet rules for substitution. Just try to substitute for an expression that appears frequently or prominently. If two different troublesome expressions appear, try to express them both in terms of some new expression.)

(a) \(\int \frac{dx}{\sqrt{1 + e^x}}\).

**Solution**: Try $u = \sqrt{1 + e^x}$, so $du = e^x dx/(2\sqrt{1 + e^x})$, so $dx/\sqrt{1 + e^x} = 2du/e^x = 2du/(u^2 - 1)$. We get

$$
\int \frac{dx}{\sqrt{1 + e^x}} = \int \frac{2du}{u^2 - 1} = \int \left( \frac{1}{u - 1} - \frac{1}{u + 1} \right) \, du = \log |u - 1| - \log |u + 1| = \log \left| \frac{u - 1}{u + 1} \right| = \log \left| \frac{\sqrt{1 + e^x} - 1}{\sqrt{1 + e^x} + 1} \right|.
$$

But in fact the absolute value signs are not needed here: $\sqrt{1 + e^x} \geq 1$ always, so we are never attempting to evaluate log at a negative argument, and we can write

$$
\int \frac{dx}{\sqrt{1 + e^x}} = \log \left( \frac{\sqrt{1 + e^x} - 1}{\sqrt{1 + e^x} + 1} \right).
$$

(b) \(\int \frac{4^x + 1}{2^x + 1} \, dx\).

**Solution**: Make the substitution $u = 2^x + 1 = e^{x \log 2} + 1$. We have

$$
du = \log 2e^{x \log 2} \, dx
$$

5
so

\[ \frac{1}{(\log 2)} \frac{1}{2^x} \, du = \frac{1}{(\log 2)} \frac{1}{u - 1} \, du, \]

and

\[ 4^x + 1 = (2^x)^2 + 1 = (u - 1)^2 + 1. \]

We get

\[ \int \frac{4^x + 1}{2^x + 1} \, dx = \frac{1}{\log 2} \int \frac{(u - 1)^2 + 1}{u(u - 1)} \, du \]

\[ = \frac{1}{\log 2} \int \left( 1 - \frac{1}{u} + \frac{1}{u(u - 1)} \right) \, du \]

\[ = \frac{1}{\log 2} \int \left( 1 - \frac{1}{u} + \frac{1}{u - 1} - \frac{1}{u} \right) \, du \]

\[ = \frac{1}{\log 2} \int \left( 1 - \frac{2}{u} + \frac{1}{u - 1} \right) \, du \]

\[ = \frac{1}{\log 2} (2u - 2 \log |u| + \log |u - 1|) \]

\[ = \frac{1}{\log 2} (2(2^x + 1) - 2 \log(2^x + 1) + \log 2^x) \]

\[ = \frac{1}{\log 2} (2(2^x + 1) - 2 \log(2^x + 1) + x \log 2) \quad \text{(alternate form)}. \]

(c)

\[ \int \frac{1}{x^2} \sqrt{\frac{x - 1}{x + 1}} \, dx. \]

**Solution:** Note that \( x \) is restricted here to be in the set \((-\infty, -1] \cup [1, \infty)\). Start with \( u = 1/x \) (so \( u \) is restricted to \([-1, 1] \setminus \{0\})\), so \( du = -dx/x^2 \), so \(-du = dx/x^2\). Also

\[ \sqrt{\frac{x - 1}{x + 1}} = \sqrt{\frac{1 - 1/x}{1 + 1/x}} = \sqrt{\frac{1 - u}{1 + u}}, \]

so

\[ \int \frac{1}{x^2} \sqrt{\frac{x - 1}{x + 1}} \, dx = - \int \sqrt{\frac{1 - u}{1 + u}} \, du. \]

We can manipulate the integrand to make it amenable to a trigonometric substitution:

\[ \int \sqrt{\frac{1 - u}{1 + u}} \, du = \int \sqrt{\frac{1 - u}{1 + u}} \sqrt{1 - u} \, du = \int \frac{1 - u}{\sqrt{1 - u^2}} \, du. \]

Via substitution \( u = \sin t \) (which is valid — \( u \) is restricted to \([-1, 1])\), \( du = \cos t \, dt \), we get

\[ \int \frac{1 - u}{\sqrt{1 - u^2}} \, du = \int \frac{1 - \sin t}{\cos t} \, \cos t \, dt = \int (1 - \sin t) \, dt = t + \cos t. \]
So
\[
\int \frac{1}{x^2} \sqrt{\frac{x-1}{x+1}} \, dx = -t - \cos t
\]
\[
= -\arcsin u - \cos(\arcsin u)
\]
\[
= -\arcsin(1/x) - \cos(\arcsin(1/x)).
\]

If \( x \geq 1 \), then by the standard right-triangle argument, \( \cos(\arcsin(1/x)) = \sqrt{1 - 1/x^2} \), and this is also valid if \( x \leq -1 \), so
\[
\int \frac{1}{x^2} \sqrt{\frac{x-1}{x+1}} \, dx = -\arcsin(1/x) - \sqrt{1 - 1/x^2}.
\]

5. Next, some integrals where it might not be too ridiculous to consider the “magic bullet” substitution \( t = \tan(x/2) \).

(a)\[
\int \frac{dx}{a \sin x + b \cos x}. \quad (a, b \text{ arbitrary constants})
\]

**Solution:** Via \( t = \tan(x/2) \) we get
\[
\int \frac{dx}{a \sin x + b \cos x} = \frac{1}{\sqrt{a^2 + b^2}} \int \frac{2dt}{t^2 + 1} = \frac{2dt}{bt^2 - b^2}.
\]

We deal first with a boundary cases of this integral. If \( b = 0 \), then the integral becomes
\[
\frac{1}{a} \int \frac{dt}{t} = \log |t| = \log |\tan(x/2)|.
\]
(In this case the integral is \((1/a) \int \csc x \, dx\), which is more traditionally presented as \( \log |\csc x - \cot x|\); the two answers are easily checked to be the same).

Otherwise, whatever the values of \( a, b \), the quadratic in the denominator always has two real roots (its discriminant (the “\( b^2 - 4ac \)” in the quadratic formula) is \( 4a^2 + 4b^2 \), which is always positive). So the denominator factors into two real roots. Specifically:
\[
b + 2at - bt^2 = -b \left( t - \frac{a + \sqrt{a^2 + b^2}}{b} \right) \left( t - \frac{a - \sqrt{a^2 + b^2}}{b} \right).
\]

A partial fractions decomposition gives:
\[
\frac{2}{b + 2at - bt^2} = \frac{1}{\sqrt{a^2 + b^2}} \left( \frac{1}{t - \frac{a - \sqrt{a^2 + b^2}}{b}} - \frac{1}{t - \frac{a + \sqrt{a^2 + b^2}}{b}} \right).
\]

The value of the integral when \( b \neq 0 \) is thus (obviously):
\[
\int \frac{dx}{a \sin x + b \cos x} = \frac{1}{\sqrt{a^2 + b^2}} \log \left| \frac{\tan(x/2) - \frac{a - \sqrt{a^2 + b^2}}{b}}{\tan(x/2) - \frac{a + \sqrt{a^2 + b^2}}{b}} \right|.
\]
(b) (A slight variant of the “magic bullet” substitution might be in order here):

\[
\int \frac{dx}{1 - \sin^2 x}.
\]

**Solution**: This was something of a sneaky question;

\[
\frac{1}{1 - \sin^2 x} = \sec^2 x,
\]

so

\[
\int \frac{dx}{1 - \sin^2 x} = \tan x.
\]

Here’s how you might arrive at this using a substitution: via \( t = \tan(x/2) \) we get

\[
\int \frac{dx}{1 - \sin^2 x} = \int \left( \frac{1}{1 - \frac{4t^2}{(1+t^2)^2}} \right) \frac{2dt}{1 + t^2}
\]

\[
= \int \frac{2(1 + t^2)}{1 - 2t^2 + t^4} dt.
\]

This is certainly do-able by partial fractions:

\[
\frac{2(1 + t^2)}{1 - 2t^2 + t^4} = \frac{1}{(t+1)^2} + \frac{1}{(t-1)^3},
\]

so the integral is

\[
\frac{-1}{t+1} + \frac{-1}{t-1} = \frac{-2t}{t^2 - 1} = \frac{-2\tan(x/2)}{\tan^2(x/2) - 1}
\]

Here is the variant that is helpful: instead set \( t = \tan x \), so \( dt = \sec^2 x \ dx \), so

\[
dx = \frac{dt}{\sec^2 x} = \frac{dt}{1 + \tan^2 x} = \frac{dt}{1 + t^2},
\]

and

\[
\frac{1}{1 - \sin^2 x} = \frac{1}{\cos^2 x} = \sec^2 x = 1 + \tan^2 x = 1 + t^2,
\]

and the integral becomes

\[
\int 1 \ dt = t = \tan x.
\]

It’s not obvious that \( \tan x = \frac{-2\tan(x/2)}{\tan^2(x/2) - 1} \), but a little fiddling shows that they are equal.

(c)

\[
\int \frac{dx}{3 + 5\sin x}.
\]
Solution: Via $t = \tan(x/2)$ we get

\[
\int \frac{dx}{3 + 5 \sin x} = \int \left( \frac{1}{3 + \frac{10t}{1+t^2}} \right) \frac{2dt}{1+t^2}
\]

\[
= \int \left( \frac{3}{4(1+3t)} - \frac{1}{4(3+t)} \right) dt
\]

\[
= \frac{3}{4} \log |1 + 3t| - \frac{1}{4} \log |3 + t|
\]

\[
= \frac{3}{4} \log |1 + 3 \tan(x/2)| - \frac{1}{4} \log |3 + \tan(x/2)|.
\]

6. Next, some integrands appropriate for partial fractions:

(a) \[
\int \frac{2x^2 + 7x - 1}{x^3 - 3x^2 + 3x - 1} \, dx.
\]

Solution: Have

\[
x^3 - 3x^2 + 3x - 1 = (x-1)^3
\]

and

\[
\frac{2x^2 + 7x - 1}{x^3 - 3x^2 + 3x - 1} = \frac{8}{(x-1)^3} + \frac{11}{(x-1)^2} + \frac{2}{x-1},
\]

so

\[
\int \frac{2x^2 + 7x - 1}{x^3 - 3x^2 + 3x - 1} \, dx = -\frac{4}{(x-1)^2} - \frac{11}{x-1} + 2 \log |x-1|.
\]

(b) \[
\int \frac{3x^2 + 3x + 1}{x^3 + 2x^2 + 2x + 1} \, dx.
\]

Solution: Have

\[
x^3 + 2x^2 + 2x + 1 = (x+1)(x^2 + x + 1)
\]

and

\[
\frac{3x^2 + 3x + 1}{x^3 + 2x^2 + 2x + 1} = \frac{2x}{x^2 + x + 1} + \frac{1}{x+1}.
\]

Now

\[
\frac{2x}{x^2 + x + 1} = \frac{2x}{(x + (1/2))^2 + (\sqrt{3}/2)^2}
\]

\[
= \frac{2(x + (1/2))}{(x + (1/2))^2 + (\sqrt{3}/2)^2} - \frac{1}{(x + (1/2))^2 + (\sqrt{3}/2)^2}.
\]
Each of the three terms
\[
\frac{1}{x+1}, \quad \frac{2(x+(1/2))}{(x+(1/2))^2 + (\sqrt{3}/2)^2}, \quad \frac{1}{(x+(1/2))^2 + (\sqrt{3}/2)^2}
\]
are readily integrable, and
\[
\int \frac{3x^2 + 3x + 1}{x^3 + 2x^2 + 2x + 1} \, dx = \log |x+1| + \log |(x+(1/2))^2 + (\sqrt{3}/2)^2| + \frac{2}{\sqrt{3}} \arctan \left( \frac{2x+1}{\sqrt{3}} \right).
\]
(For the last integral:
\[
\int \frac{du}{u^2 + a^2} = \frac{1}{a} \tan^{-1} \left( \frac{u}{a} \right)
\]
is easily derived from \( \int dx/(x^2 + 1) = \tan^{-1} x \).)

(c)
\[
\int \frac{3x}{(x^2 + x + 1)^3} \, dx.
\]

**Solution:** The integral is already in the correct form for partial fractions. We write
\[
\frac{3x}{(x^2 + x + 1)^3} = \frac{3(x+(1/2))}{(x+(1/2))^2 + (\sqrt{3}/2)^2} - \frac{3/2}{(x+(1/2))^2 + (\sqrt{3}/2)^2},
\]
and use reduction formulae from class to get:
\[
\int \frac{3x}{(x^2 + x + 1)^3} \, dx = - \left( \frac{2x^3 + 3x^2 + 4x + 3}{2(x^2 + x + 1)^2} \right) + \frac{2}{\sqrt{3}} \arctan \left( \frac{2x+1}{\sqrt{3}} \right).
\]

7. Next, a pot-pourri with a (slightly non-obvious) trigonometric flavor.

(a)
\[
\int \sqrt{1 - 4x - 2x^2} \, dx.
\]

**Solution:** First complete the square:
\[
1 - 4x - 2x^2 = -2(x^2 + 2x - 1/2) = -2((x+1)^2 - 3/2) = 3 - 2(x+1)^2 = (\sqrt{3})^2 - (\sqrt{2}(x+1)^2).
\]
Now make substitution
\[
\frac{\sqrt{2}}{\sqrt{3}}(x + 1) = \sin t.
\]
So
\[
\sqrt{(\sqrt{3})^2 - (\sqrt{2}(x+1)^2)} = \sqrt{3} \sqrt{1 - \sin^2 t} = \sqrt{3} \cos t,
\]
and
\[
dx = \frac{\sqrt{3}}{\sqrt{2}} \cos t \, dt.
\]
\[ \int \sqrt{1 - 4x - 2x^2} \, dx = \frac{3}{\sqrt{2}} \int \cos^2 t \, dt = \frac{3}{2\sqrt{2}} \left( t + \frac{\sin 2t}{2} \right). \]

Now
\[ \frac{\sin 2t}{2} = \sin t \cos t = \frac{\sqrt{2}}{\sqrt{3}}(x + 1) \cos t. \]

If we are in the regime where \( x + 1 \geq 0 \), so \( t \) is an angle between 0 and \( \pi/2 \), then from a right-angled triangle we get
\[ \cos t = \frac{\sqrt{1 - 4x - 2x^2}}{\sqrt{3}}. \]

So, for \( x + 1 \geq 0 \),
\[ \int \sqrt{1 - 4x - 2x^2} \, dx = \frac{3}{2\sqrt{2}} \left( \arcsin \left( \frac{\sqrt{2} (x + 1)}{\sqrt{3}} \right) + \frac{\sqrt{2}}{\sqrt{3}} (x + 1) \frac{\sqrt{1 - 4x - 2x^2}}{\sqrt{3}} \right) \]
\[ = \frac{3}{2\sqrt{2}} \arcsin \left( \frac{\sqrt{2} (x + 1)}{\sqrt{3}} \right) + \frac{(x + 1)}{2} \sqrt{1 - 4x - 2x^2}. \]

This also works when \( x + 1 \leq 0 \), so is the final answer.

(b)
\[ \int \cos x \sqrt{9 + 25 \sin^2 x} \, dx. \]

**Solution:** A natural substitution is \( \sin x = \frac{3}{5} \tan t \). Then
\[ \sqrt{9 + 25 \sin^2 x} = 3 \sqrt{1 + \tan^2 t} = 3 \sec t, \]
and
\[ \cos x \, dx = \frac{3}{5} \sec^2 t \, dt, \]
so
\[ \int \cos x \sqrt{9 + 25 \sin^2 x} \, dx = \frac{9}{5} \int \sec^3 t \, dt \]
\[ = \frac{9}{10} (\sec t \tan t + \log |\sec t + \tan t|) \]
(using a previous result on \( \int \sec^3 w \, dw \)).

Assuming we are in the regime where \( t \) is an angle between 0 and \( \pi/2 \), since we have
\[ \tan t = \frac{5 \sin x}{3}, \]
so from a right-angled triangle we get that
\[ \sec t = \frac{\sqrt{9 + 25 \sin^2 x}}{3}. \]
and so
\[
\int \cos x \sqrt{9 + 25 \sin^2 x} \, dx = \frac{9}{10} \left( \frac{5 \sin x \sqrt{9 + 25 \sin^2 x}}{9} + \log \left| \frac{\sqrt{9 + 25 \sin^2 x}}{3} + 5 \sin x \right| \right).
\]

Since there is no potential complication with this function in terms of differentiating in other regimes of \( x \), this is the answer for all \( x \).

(c) \[
\int e^{4x} \sqrt{1 + e^{2x}} \, dx.
\]

**Solution:** Start with \( u = 1 + e^{2x} \), so \( du = 2e^{2x} \, dx \), and \( dx = du / (2(u - 1)) \). Also \((u - 1)^2 = e^{4x} \). We get
\[
\int e^{4x} \sqrt{1 + e^{2x}} \, dx = \frac{1}{2} \int (u - 1) \sqrt{u} \, du
\]
\[
= \frac{u^{5/2}}{5} - \frac{u^{3/2}}{3}
\]
\[
= \frac{(1 + e^{2x})^{5/2}}{5} - \frac{(1 + e^{2x})^{3/2}}{3}.
\]

8. Finally, another pot-pourri. Who knows what methods might be needed?

(a) \[
\int \frac{x \arctan x}{(1 + x^2)^3} \, dx.
\]

**Solution:** Via integration by parts, with
\[
u = \arctan x, \quad du = \frac{dx}{1 + x^2},
\]
\[
dv = \frac{x}{(1 + x^2)^3} \, dx, \quad v = -\frac{1}{4(1 + x^2)^2},
\]
get
\[
\int \frac{x \arctan x}{(1 + x^2)^3} \, dx = -\arctan x \frac{dx}{4(1 + x^2)^2} + \int \frac{dx}{4(1 + x^2)^3}
\]
\[
= -\arctan x \frac{1}{4(1 + x^2)^2} + \frac{1}{32} \left( x(5 + 3x^2) + 3 \arctan x \right).
\]

(This last integral can be obtained from a reduction formula that we derived in class).

(b) \[
\int \log \sqrt{1 + x^2} \, dx.
\]
Solution: Via integration by parts, with

\[ u = \log \sqrt{1 + x^2}, \quad du = \frac{xdx}{1 + x^2}, \]

\[ dv = dx, \quad v = x, \]

get

\[
\int \log \sqrt{1 + x^2} \, dx = x \log \sqrt{1 + x^2} - \int \frac{x^2 dx}{1 + x^2} = x \log \sqrt{1 + x^2} - \int \left(1 - \frac{1}{1 + x^2}\right) \, dx = x \log \sqrt{1 + x^2} - x - \arctan x.
\]

(c) \[
\int \sqrt{\tan x} \, dx.
\]

Solution: Substituting \( u = \sqrt{\tan x} \),

\[ du = \frac{\sec^2 x \, dx}{2\sqrt{\tan x}} = \frac{(\tan^2 x + 1) \, dx}{2u} = \frac{u^4 + 1}{} \, dx, \]

so

\[ dx = \frac{2u}{u^4 - 1} \, du, \]

and

\[
\int \sqrt{\tan x} \, dx = \int \frac{2u^2}{u^4 - 1} \, du.
\]

Factoring \( u^4 - 1 = (u^2 + \sqrt{2}u + 1)(u^2 - \sqrt{2}u + 1) \), and writing

\[ \frac{2u}{u^4 - 1} = -\sqrt{2} \left( \frac{u}{u^2 + \sqrt{2}u + 1} - \frac{u}{u^2 - \sqrt{2}u + 1} \right), \]

after some (lots of) painful algebra, we get to

\[
\int \sqrt{\tan x} \, dx = -\frac{\sqrt{2}}{4} \log \left| \tan x + \sqrt{2} \tan x + 1 \right| + \frac{\sqrt{2}}{4} \log \left| \tan x - \sqrt{2} \tan x + 1 \right| + \frac{\sqrt{2}}{2} \arctan \left( \sqrt{2} \tan x + 1 \right) - \frac{\sqrt{2}}{2} \arctan \left( -\sqrt{2} \tan x + 1 \right)
\]

(obviously).