Math 10860, Honors Calculus 2

Homework 8  NAME:

Due in class Friday April 5

Instructions

The usual rules apply. Here’s a summary:

Please present answers neatly and clearly. Make use of space to increase the clarity of your presentation. Justify non-obvious assertions — the homework is as much about showing me that you are mastering the topics of the course, as it is about getting the right answers.

Be careful with the logical flow of your proof-based answers. Make sure that each statement you write fits in to the proof in a clear way — either as something which follows from previous statements, or whose truth would be enough to establish the truth of the result you are being challenged to prove. Use connective phrases (like “from this it follows that”, or “it is now enough to prove ..., which we now do”, et cetera), to highlight the flow of the proof.

Consider submitting your answers, to at least some of the questions, in LaTeX. I’ll make the LaTeX source of the homework available to you to get you started.

Reading for this homework

Chapter 15 of the class notes, and/or Chapter 20 of Spivak; for the questions on sequences, Chapter 22 of Spivak, and the (as-yet-unwritten) Chapter 16 of the class notes.

Assignment

1. (a) Find the Taylor polynomial of degree $2n$ of $\cos$, at $a = \pi$.
   
   (b) Find the Taylor polynomial of degree 4 of $f(x) = x^5 + x^3 + x$, at $a = 1$.
   
   (c) Express the polynomial $p(x) = ax^4 + bx^3 + cx^2 + dx + e$ as a polynomial in $(x - 2)$.

2. An important Taylor polynomial that we did not discuss much in class is that of $\log x$, at $a = 1$ (we can’t choose $a = 0$ here, since 0 is not in the domain of log). Actually, it’s nicer to consider the function $\log(1 + x)$ at $a = 0$.

   (a) By calculating derivatives, find the Taylor polynomial of degree $n$ of $\log(1 + x)$ about $a = 0$.

   (b) Show that for $-1 < x \leq 1$ the remainder term $R_{n,0,\log(1+)}(x)$ goes to zero as $n$ goes to infinity. **Hint:** It might be better to avoid the Lagrange or integral forms
of the remainder term, instead starting with
\[ \log(1 + x) = \int_0^x \frac{dt}{1 + t}. \]

(c) Use Taylor polynomials, and your analysis of the remainder term, to find a rational number that is within \( \pm 0.1 \) of \( \log 2 \).

(d) Show that for \( x > 1 \) the remainder term \( R_{n,0,\log(1 + \cdot)}(x) \) does not go to zero as \( n \) goes to infinity.

(e) Nevertheless, use Taylor polynomials (slightly cleverly) to find a rational number that is within \( \pm 0.1 \) of \( \log 3 \).

3. Here is (something of) a generalization of the binomial theorem. Recall that the binomial theorem says that for all natural numbers \( n \), and for all real \( x \),
\[ (1 + x)^n = 1 + \binom{n}{1} x + \binom{n}{2} x^2 + \cdots + \binom{n}{n} x^n, \]
where for natural numbers \( k \) and \( n \), \( \binom{n}{k} = \frac{n(n-1)\cdots(n-(k-1))}{k!} \).

For an arbitrary real number \( \alpha \), and natural number \( k \), define
\[ \binom{\alpha}{k} = \frac{\alpha(\alpha - 1)\cdots(\alpha - (k - 1))}{k!} \]
(note that this agrees with \( \binom{n}{k} \) when \( \alpha = n \)).

Let \( f_\alpha : (-1, \infty) \to \mathbb{R} \) be defined by \( f_\alpha(x) = (1 + x)^\alpha \).

(a) Show that the Taylor polynomial of degree \( n \) of \( f_\alpha \) about 0 is
\[ P_{n,0,f_\alpha}(x) = 1 + \binom{\alpha}{1} x + \binom{\alpha}{2} x^2 + \cdots + \binom{\alpha}{n} x^n, \]
and that the remainder term can be expressed as \( R_{n,0,f_\alpha}(x) = (\frac{\alpha}{n+1}) x^{n+1}(1 + t)^{\alpha-n-1} \) for some \( t \) between 0 and \( x \).

(b) The remainder term above is quite difficult to pin down (we will consider it in class in a few weeks). In some special cases, though, it is reasonable.

i. Show that
\[ \binom{-1/2}{n+1} = (-1)^{n+1} \frac{(2n+2)}{2^{2n+2}} \]
(this requires level-headed algebraic manipulation. It helps to know what you are aiming for in advance!).

ii. Deduce that for \( 0 < x < 1 \), \( R_{n,0,f_{-1/2}}(x) \to 0 \) as \( n \) grows, and that for \( -1/2 < x < 0 \), \( R_{n,0,f_{-1/2}}(x) \to 0 \) as \( n \) grows.
4. Find the following limits:

(a) \( \lim_{n \to \infty} \frac{n}{n+1} \). (For this one, you must use the definition of sequence limit).

(b) \( \lim_{n \to \infty} \sqrt{n^2 + n} \). (For this and the remaining parts, a soft argument is fine).

(c) \( \lim_{n \to \infty} \left( \sqrt{n^2 + 1} - \sqrt{n + 1} \right) \).

(d) \( \lim_{n \to \infty} \left( \frac{n}{n+1} - \frac{n+1}{n} \right) \).

(e) \( \lim_{n \to \infty} \frac{\sqrt{n^2}}{n!} \).

(f) \( \lim_{n \to \infty} \frac{(-1)^n \sqrt{n \sin(n)}}{n+1} \).

5. Consider the sequence

\[
\left( \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{6}, \ldots \right)
\]

For which numbers \( \alpha \) is there a subsequence converging to \( \alpha \)?

**An extra-credit problem:** A sequence \((a_n)\) of non-negative terms is subadditive if \(a_{n+m} \leq a_n + a_m\) for all \(n, m \geq 1\). Prove that for a subadditive sequence of non-negative terms, the sequence \(\left( \frac{a_n}{n} \right)\) is bounded below and converges to \( \inf \{ \frac{a_n}{n} : n \in \mathbb{N} \} \).