Instructions

The usual rules apply. Here’s a summary:

Please present answers neatly and clearly. Make use of space to increase the clarity of your presentation. Justify non-obvious assertions — the homework is as much about showing me that you are mastering the topics of the course, as it is about getting the right answers.

Be careful with the logical flow of your proof-based answers. Make sure that each statement you write fits in to the proof in a clear way — either as something which follows from previous statements, or whose truth would be enough to establish the truth of the result you are being challenged to prove. Use connective phrases (like “from this it follows that”, or “it is now enough to prove ..., which we now do”, et cetera), to highlight the flow of the proof.

Consider submitting your answers, to at least some of the questions, in LaTeX. I’ll make the LaTeX source of the homework available to you to get you started.

Reading for this homework

Chapters 16 and 17 of the class notes.

Assignment

1. (a) Prove that if $0 < a < 2$ then $a < \sqrt{2a} < 2$.

   (b) Prove that the sequence

   \[
   \sqrt{2}, \sqrt{2\sqrt{2}}, \sqrt{2\sqrt{2\sqrt{2}}}, \sqrt{2\sqrt{2\sqrt{2\sqrt{2}}}}, \ldots
   \]

   converges.

   (c) Let $a_n$ be the $n$th term of the above sequence, and let $\ell = \lim_{n \to \infty} a_n$. Carefully applying a theorem we proved in class, find $\ell$.

2. Identify the function

   \[f(x) = \lim_{n \to \infty} \left( \lim_{k \to \infty} (\cos(n!\pi x))^{2k} \right).
   \]

   (It should be a very familiar function).
3. This question provides a useful estimate on $n!$: $n! \approx (n/e)^n$.

(a) Show that if $f : [1, \infty)$ is increasing then
$$f(1) + \cdots + f(n - 1) < \int_1^n f(x)dx < f(2) + \cdots + f(n).$$

(b) By taking $f = \log$ deduce that
$$\frac{n^n}{e^{n-1}} < n! < \frac{(n + 1)^{n+1}}{e^n}.$$ 

(c) Deduce that\footnote{Note that this only says that for large $n$, $\sqrt[n]{n!}$ is close to $n/e$; it does not say that for large $n$, $n!$ is close to $(n/e)^n$ — it is not. In fact, all we can get out of the bounds in part b) is that $e \left(\frac{n}{e}\right)^n < n! < e(n + 1) \left(\frac{n}{e}\right)^n$.}
$$\lim_{n \to \infty} \frac{\sqrt[n]{n!}}{n} = \frac{1}{e}.$$ 

4. The Harmonic number $H_n$ is the number $H_n = \sum_{k=1}^{n} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$. Recall that the sequence $(H_n)_{n=1}^{\infty}$ does not converge. This exercise gives a very useful estimate on $H_n$, namely $H_n \approx \log n$.

(a) Prove that for all natural numbers $n$,
$$\frac{1}{n + 1} < \log(n + 1) - \log n < \frac{1}{n}.$$ 

(b) Deduce from part (a) that the sequence $(H_n - \log n)_{n=2}^{\infty}$ is decreasing and bounded below by $0$.\footnote{From this it follows that there is a non-negative number, traditionally denoted $\gamma$, such that $\lim_{n \to \infty} (H_n - \log n) = \gamma$. This number is known as the Euler-Mascheroni constant, and is approximately $0.57721$. It is not known whether $\gamma$ is rational or irrational.}

5. Decide whether the following sums converge.

A better, and much more difficult to prove, bound on $n!$ is given by Stirling’s formula:
$$\lim_{n \to \infty} \frac{n!}{\sqrt{2\pi n} \left(\frac{n}{e}\right)^n};$$
in other words, for all $\varepsilon > 0$ there is $n_0$ such that $n > n_0$ implies
$$(1 - \varepsilon)\sqrt{2\pi n} \left(\frac{n}{e}\right)^n < n! < (1 + \varepsilon)\sqrt{2\pi n} \left(\frac{n}{e}\right)^n.$$
6. (a) Show that
\[ \sum_{n=1}^{\infty} (-1)^n \frac{\log n}{n} \]
converges for \(0 < a < e\) and diverges for \(a > e\).

(b) When \(a = e\), show that the series diverges, by using the a result proved earlier in this homework.

(c) Decide when
\[ \sum_{n=1}^{\infty} \frac{a^n n!}{n^n} \]
converges, again using a previous result on this homework when the ratio test fails.

7. (a) Prove that if \(a_n \geq 0\) and \((a_n)\) is not summable (i.e., \(\sum a_n\) diverges), then \((a_n/(1 + a_n))\) is not summable.

(b) Is the converse true? If \((a_n/(1 + a_n))\) is not summable (with \(a_n > 0\)), must it always be the case that \((a_n)\) is not summable?

8. Wally, a slow but persistent worm, starts at one end of a meter-long rubber band and crawls one centimeter per minute toward the other end. At the end of each minute Wally rests for a moment. During that moment Karl, the equally persistent keeper of the band (whose sole purpose in life is to frustrate Wally) stretches the band one meter. Thus after one minute of crawling, Wally is 1 centimeter from the start and 99 from the finish; but in the moment that Wally rests, Karl stretches the band one meter. During the stretching Wally maintains his relative position, 1% from the start and 99% from the finish. So at the end of Wally’s moment of rest, he is 2cm from the starting point and 198cm from his goal. After Wally crawls for another minute the score is 3cm traveled and 197 to go; but then Karl stretches one more meter (from 2 meters to 3 meters), and Wally’s distances become 4.5cm travelled, and 295.5cm to go. And so on.

Does Wally ever get to the end of the band???

He keeps moving, but the goal seems to be moving away from him, faster than he moves. (We’re assuming here infinite longevity for Karl and Wally, infinite elasticity of the band, and an infinitely tiny worm.)
An extra-credit problem: Define the 7-depleted harmonic number $H^{(7)}_n$ to be the sum of the reciprocals of the natural numbers from 1 to $n$, except those $n$ that have a 7 in their decimal expansion. For example, $H^{(7)}_8 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{8}$. (There is no standard name or notation for this number).

Does $(H^{(7)}_n)_{n=1}^{\infty}$ converge or diverge?