Math 10860, Honors Calculus 2

Midterm 2 practice problems solutions

Spring 2019

1. (a) Express $\sin x - \sin y$ as a product of two trigonometric functions.

Solution: We have

$$\sin(a + b) = \sin a \cos b + \cos a \sin b$$

and

$$\sin(a - b) = \sin a \cos b - \cos a \sin b$$

so

$$\sin(a + b) - \sin(a - b) = 2 \cos a \sin b.$$ Setting $x = a + b$ and $y = a - b$, get $a = (x + y)/2$ and $b = (x - y)/2$, so

$$\sin x - \sin y = 2 \cos \left( \frac{x + y}{2} \right) \sin \left( \frac{x - y}{2} \right).$$

(b) Express $\sin \left( k + \frac{1}{2} \right) x - \sin \left( k - \frac{1}{2} \right) x$ as a product of two trigonometric functions.

Solution: Apply the previous result with the role of $x$ played by $(k + 1/2)x$ and the role of $y$ played by $(k - 1/2)x$. We get

$$\sin \left( k + \frac{1}{2} \right) x - \sin \left( k - \frac{1}{2} \right) x = 2 \cos kx \sin \frac{x}{2}.$$ 

(c) Prove that for natural numbers $n$,

$$\cos x + \cos 2x + \cdots + \cos nx = \frac{\sin \left( n + \frac{1}{2} \right) x}{2 \sin \frac{x}{2}} - \frac{1}{2}.$$ 

Solution: We have, using the result of the last part for the first line, and the fact that the sum is telescoping for the second,

$$2 \cos x \sin \frac{x}{2} + \cdots + 2 \cos nx \sin \frac{x}{2} = \sum_{k=1}^{n} \left( \sin \left( k + \frac{1}{2} \right) x - \sin \left( k - \frac{1}{2} \right) x \right)$$

$$= \sin \left( n + \frac{1}{2} \right) x - \sin \frac{x}{2}.$$ 

Dividing both sides by $2 \sin \frac{x}{2}$ gives the claimed identity.
2. Prove the following two equalities: for all \( n \geq 0 \) we have

\[
\cos n\theta = \sum_{k \geq 0, \ k \ \text{even}} (-1)^{\frac{k}{2}} \binom{n}{k} \cos^{n-k} \theta \sin^k \theta
\]

and

\[
\sin n\theta = \sum_{k \geq 1, \ k \ \text{odd}} (-1)^{\frac{k-1}{2}} \binom{n}{k} \cos^{n-k} \theta \sin^k \theta.
\]

**Solution:** The most straightforward approach is probably a *joint* induction on \( n \), proving the predicate \( P(n) \) for \( n \geq 0 \), where \( P(n) \) is precisely

\[
\cos n\theta = \sum_{k \geq 0, \ k \ \text{even}} (-1)^{\frac{k}{2}} \binom{n}{k} \cos^{n-k} \theta \sin^k \theta
\]

and

\[
\sin n\theta = \sum_{k \geq 1, \ k \ \text{odd}} (-1)^{\frac{k-1}{2}} \binom{n}{k} \cos^{n-k} \theta \sin^k \theta.
\]

It is necessary to take *both* of the equalities together in a proof by induction, because the obvious argument (for the induction step) used to prove the first equality starts with

\[
\cos n\theta = \cos \theta \cos (n-1)\theta - \sin \theta \sin (n-1)\theta,
\]

and immediately an earlier case of the second equality is needed to continue; and the obvious argument (for the induction step) used to prove the second equality starts with

\[
\sin n\theta = \sin \theta \cos (n-1)\theta + \cos \theta \sin (n-1)\theta,
\]

and again, immediately an earlier case of the second equality is needed to continue.

The induction itself is a little messy, involving a good deal of somewhat annoying algebra, but it is fairly straightforward and I won’t write it down here. The key point is to gather like terms in the summation and use Pascal’s identity to combine binomial coefficients.

The base case for the induction is \( n = 0 \). The identity for \( \cos \) is easy here, but the one for \( \sin \) may initial seem confusing, as at \( n = 0 \) there are no terms in the sum on the right-hand side. We need to use the convention (mentioned once in the fall, but I don’t think ever seen “in the wild” until now) that an empty sum evaluates to 0.

3. (a) State the formula for integration by parts, and state hypotheses on the functions involved under which the formula is valid.

**Solution:** for functions \( f, g \) for which \( f', g' \) both continuous,

\[
\int f(x)g'(x) \ dx = f(x)g(x) - \int f'(x)g(x) \ dx.
\]
(b) Let \( I_n = \int x^n \sin x \, dx \).

i. Find \( I_0 \) and \( I_1 \).

**Solution:** \( I_0 = \int x \, dx = -\cos x \), while (via integration by parts)

\[
I_1 = \int x \sin x \, dx = -x \cos x + \int \cos x \, dx = -x \cos x + \sin x.
\]

ii. Find a reduction formula that expresses \( I_{n+2} \) in terms of \( I_n \) for \( n \geq 0 \).

**Solution:** We have, doing integration by parts twice,

\[
I_{n+2} = \int x^{n+2} \sin x \, dx
\]

\[
= -x^{n+2} \cos x + (n+2) \int x^{n+1} \cos x \, dx
\]

\[
= -x^{n+2} \cos x + (n+2) \left[ x^{n+1} \sin x - (n+1) \int x^n \sin x \, dx \right]
\]

\[
= -x^{n+2} \cos x + (n+2)x^{n+1} \sin x - (n+2)(n+1)I_n.
\]

4. Find the following integrals by an appropriate substitution:

(a) \[
\int \frac{\sqrt{1-x}}{1-\sqrt{x}} \, dx.
\]

**Solution:** Start with \( u = \sqrt{x} \), so \( du = (1/2\sqrt{x})dx = (1/2u)dx \), so \( dx = 2u \, du \). Also \( 1 - x = 1 - u^2 \), so integral becomes

\[
2 \int \frac{u\sqrt{1-u^2}}{1-u} \, du.
\]

Now try \( u = \sin t \), so \( du = \cos t \, dt \), to get

\[
2 \int \frac{\sin t \cos^2 t}{1-\sin t} \, dt = 2 \int \frac{\sin t(1-\sin t)(1+\sin t)}{1-\sin t} \, dt = 2 \int (\sin t + \sin^2 t) \, dt
\]

The value of this last is \(-2 \cos t + t - \frac{\sin t}{2}\). Going back to \( u \), get

\[
-2 \cos \sin^{-1} u + \sin^{-1} u - \frac{u}{2} = -2\sqrt{1-u^2} + \sin^{-1} u - \frac{u}{2}.
\]

Going back to \( x \), get

\[
-2\sqrt{1-x} + \sin^{-1} \sqrt{x} - \frac{\sqrt{x}}{2}.
\]
\[ \int \frac{dx}{x\sqrt{1+x^2}}. \]

**Solution:** Let \( x = \tan u \), so \( dx = \sec^2 u \, du \), and \( \sqrt{1+x^2} = \sec x \), so the integral becomes
\[ \int \frac{\sec^2 u}{\tan u \sec u} \, du = \int \csc u \, du. \]
This evaluates to \(-\log |\csc u + \cot u|\), or, when re-expressed in terms of \( x \),
\[ -\log \left| \frac{1}{x} + \frac{\sqrt{1+x^2}}{x} \right| = \log \left| \frac{x}{1 + \sqrt{1+x^2}} \right|. \]

5. Recall that \( \sinh x = \frac{e^x - e^{-x}}{2} \).

(a) Find the degree \( 2n+1 \) Taylor polynomial of \( \sinh \) about 0.

**Solution:** The \( k \)th derivative of \( (e^x)/2 \) at 0 is \( 1/2 \), and the \( k \)th derivative of \( (e^{-x})/2 \) at 0 is \( (-1)^k/2 \). So the \( k \)th derivative of \( \sinh x \) at 0 is
- 1 if \( k \) is odd,
- 0 if \( k \) is even.

We get
\[ P_{2n+1,0,\sinh}(x) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots + \frac{x^{2n+1}}{(2n+1)!}. \]

(b) Write down the Lagrange form of the remainder term \( R_{2n+1,0,\sinh}(x) \).

**Solution:** Directly from Taylor’s theorem,
\[ R_{2n+1,0,\sinh}(x) = \frac{\sinh^{(2n+2)}(c)x^{2n+2}}{(2n+2)!} = \frac{(e^c - e^{-c})x^{2n+2}}{2(2n+2)!} \]
for some number \( c \) between 0 and \( x \).

(c) Show that for all real \( x \) the remainder term \( R_{2n+1,0,\sinh}(x) \) tends to 0 as \( n \) tends to infinity.

**Solution:** Fix a real number \( x \). The function \( (e^x - e^{-x})/2 \) is increasing on its domain (all reals), so
\[ \frac{(e^c - e^{-c})}{2} \leq \max \left\{ \frac{(e^x - e^{-x})}{2}, \frac{(e^0 - e^{-0})}{2} = 0 \right\}. \]
So
\[ \left| \frac{(e^c - e^{-c})}{2} \right| \leq \left| \frac{(e^x - e^{-x})}{2} \right|. \]
Whatever this is, it is just some constant \( C_x \) (depending on \( x \)). Thus we have
\[ |R_{2n+1,0,\sinh}(x)| \leq C_x \frac{|x|^{2n+2}}{(2n+2)!}. \]
We have proven that for all \( x > 0 \), \( \lim_{n \to \infty} \frac{x^n}{n!} = 0 \), so
\[
|R_{2n+1,0,\sinh(x)}| \to 0
\]
as \( n \to \infty \), as required.

6. Set \( a_1 = 1 \) and for \( n \geq 1 \), set
\[
a_{n+1} = \sqrt{6 + a_n}.
\]

(a) Prove that \( (a_n)_{n=1}^{\infty} \) is bounded from above. **Hint:** It will be helpful to figure out what \( \lim_{n \to \infty} a_n \) ought to be, before tackling this part.

**Solution:** Tackling this and the next question together, we claim that
\[
a_n \leq \frac{1 + \sqrt{1 + 4p}}{2}
\]
for all \( n \) (bound is 3 when \( p = 6 \)). We prove this by induction on \( n \); base case \( n = 1 \) is easy algebra (using \( p \geq 0 \)). For the induction step,
\[
a_{n+1} \leq \frac{1 + \sqrt{1 + 4p}}{2} \text{ if and only if } \sqrt{p + a_n} \leq \frac{1 + \sqrt{1 + 4p}}{2}
\]
if and only if
\[
4(p + a_n) \leq (1 + \sqrt{1 + 4p})^2
\]
if and only if
\[
4p + 4a_n \leq 2 + 4p + 2\sqrt{1 + 4p}
\]
if and only if
\[
a_n \leq \frac{1 + \sqrt{1 + 4p}}{2},
\]
so we have our claimed upper bound by induction.

(b) Prove that \( (a_n)_{n=1}^{\infty} \) is non-decreasing.

**Solution:** For each \( n \) we have
\[
a_{n+1} \geq a_n \text{ if and only if } \sqrt{p + a_n} \geq a_n
\]
if and only if
\[
p + a_n \geq a_n^2
\]
if and only if
\[
a_n^2 - a_n - p \leq 0.
\]
It’s an easy algebra check that when \( a_n \leq \frac{1 + \sqrt{1 + 4p}}{2} \) (and \( a_n \geq 0 \), as it obviously is), then \( a_n^2 - a_n - p \leq 0 \); and since we know \( a_n \leq \frac{1 + \sqrt{1 + 4p}}{2} \) (from a previous part), we get that \( a_{n+1} \geq a_n \).

(c) Explain why \( (a_n)_{n=1}^{\infty} \) converges to a limit \( \ell \) (just state the result we have proven that allows this to be concluded), and calculate \( \ell \) (with brief justification).

**Solution:** We know that \( (a_n) \) converges to a limit because it is non-decreasing and bounded above.

We claim that the limit is \( \frac{1 + \sqrt{1 + 4p}}{2} \). Indeed, suppose that the limit is \( \ell \), which is definitely some non-negative number. We have
\[
(a_n) \to \ell
\]
so, with $f$ the function $f(x) = \sqrt{p + x}$ (definitely continuous at $\ell$), we have

$$(f(a_n)) \to f(\ell) = \sqrt{p + \ell}.$$ 

But $f(a_n) = a_{n+1}$, and so since $(a_{n+1}) \to \ell$, we have

$$(f(a_n)) \to \ell.$$ 

By uniqueness of limits, we have

$$\sqrt{p + \ell} = \ell$$

which (after some algebra, and noting that $\ell \geq 0$), reduces to

$$\ell = \frac{1 + \sqrt{1 + 4p}}{2},$$

as claimed.

7. The last question shows that the expression

$$\sqrt{6 + \sqrt{6 + \sqrt{6 + \cdots}}}$$

makes sense. Re-do it with “6” replace by “$p$”, for arbitrary (but fixed) $p \geq 0$. That is, show that

$$\sqrt{p + \sqrt{p + \sqrt{p + \cdots}}}$$

makes sense, and calculate its value.

**Solution:** See last question.