1. (a) Give a clear & correct statement of the fundamental theorem of calculus, part 1.

**Solution:** If $I$ is an interval, and $f : I \to \mathbb{R}$ is integrable on (every closed interval contained in) $I$, and the function $F : I \to \mathbb{R}$ is defined by $F(x) = \int_a^x f$ (or, $F(x) = \int_a^x f(t) \, dt$), and if $f$ is continuous at $c$, then $F$ is differentiable at $c$, and $F'(c) = f(c)$.

A corollary of this is that if $f$ is continuous on $I$ then $F$ is differentiable on $I$, and $F' = f$ on $I$, but this corollary doesn’t imply FTOC1 — knowing something about what happens if $f$ is continuous everywhere doesn’t allow us to deduce anything about what happens if $f$ is continuous at just one point. So this cannot be considered a correct answer to the question.

(b) Give a clear & correct statement of the fundamental theorem of calculus, part 2.

**Solution:** If $f : [a, b] \to \mathbb{R}$ is integrable, and if there is a function $g : [a, b] \to \mathbb{R}$ satisfying $g' = f$ on $[a, b]$, then $\int_a^b f = g(b) - g(a)$.

The weaker statement that replaces “$f$ is integrable” with “$f$ is continuous” is not quite FTOC2; it is really just a corollary of FTOC1.

2. Determine directly (without comparison to other, known, integrals) whether the improper integral $\int_1^\infty \frac{dx}{\sqrt{x}}$ exists. (You may use FTOC if you wish.)

**Solution:** The easiest approach uses FTOC2. On $[1, N]$ and antiderivative for $f$ is $g(x) = 2\sqrt{x}$, so $\int_1^N f = 2\sqrt{N} - 2\sqrt{1}$. Since $2\sqrt{N} - 2\sqrt{1}$ can be made arbitrarily large by choosing $N$ large enough, we get that $\lim_{N \to \infty} \int_1^N \frac{dx}{\sqrt{x}}$ does not exist.

A longer, but more direct approach, combines elements of our proof that $1/x$ is not integrable on $[1, \infty]$, with elements of the proof of the comparison theorem. We have that $1/\sqrt{x}$ is decreasing on $[1, \infty)$, and is continuous, so for $1 \leq a < b < \infty$,

$$\int_a^b \frac{dx}{\sqrt{x}} \geq \frac{b - a}{\sqrt{b}}.$$

In particular,

$$\int_1^n \frac{dx}{\sqrt{x}} \geq \frac{n - 1}{\sqrt{n}} = \sqrt{n} - \frac{1}{\sqrt{n}} \geq \frac{\sqrt{n}}{2},$$

with the last inequality valid for all sufficiently large $n$ (specifically, it is valid for $n \geq 2$).

Now since $\sqrt{n}/2 \to \infty$ as $n \to \infty$, it follows that $\int_1^\infty \frac{dx}{\sqrt{x}}$ does not exist.