1. State Taylor’s Theorem with the Lagrange form of the remainder term.

**Solution:** If \( f \) is a function with \( f, f', f'', \ldots, f^{(n+1)} \) all existing on some interval that includes both \( a \) and \( x \), then there is some number \( c \) between \( a \) and \( x \) such that

\[
f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}.
\]

(or: \( f(x) = P_{n,a,f}(x) + R_{n,a,f}(x) \), where \( R_{n,a,f}(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1} \).

**Note:** When we proved this, we added the hypothesis that \( f^{(n+1)} \) is continuous on the interval whose endpoints are \( a \) and \( x \), and it’s ok to state the theorem in those terms.

2. (a) Let \( f(x) = \frac{1}{x^2} \). By repeatedly calculating derivatives, write down the Taylor polynomial of degree \( n \) of \( f \) at 0, that is, \( P_{n,0,f}(x) \). (You don’t need to do this formally by induction; once you have spotted the pattern, go with it.)

**Solution:** We have \( f(0) = 1 \); \( f'(x) = 1/(1-x)^2 \), so \( f'(0) = 1 \); \( f''(x) = 2/(1-x)^3 \), so \( f''(0) = 2 \); \( f'''(x) = 6/(1-x)^4 \), so \( f'''(0) = 6 \); and in general if \( f^{(k)}(x) = k!/(1-x)^{k+1} \), so \( f^{(k)}(0) = k! \), then \( f^{(k+1)}(x) = (k+1)!/(1-x)^{k+2} \), so \( f^{(k+1)}(0) = (k+1)! \). So

\[
P_{n,0,f}(x) = 1 + x + x^2 + \cdots + x^n.
\]

The quiz as given out asked for \( P_{n,a,f}(x) \) We have \( f(a) = 1/(1-a) \); \( f'(x) = 1/(1-x)^2 \), so \( f'(a) = 1/(1-a)^2 \); \( f''(x) = 2/(1-x)^3 \), so \( f''(a) = 2/(1-a)^3 \); \( f'''(x) = 6/(1-x)^4 \), so \( f'''(a) = 6/(1-a)^4 \); and in general if \( f^{(k)}(x) = k!/(1-x)^{k+1} \), so \( f^{(k)}(a) = k!/(1-a)^{k+1} \), then \( f^{(k+1)}(x) = (k+1)!/(1-x)^{k+2} \), so \( f^{(k+1)}(a) = (k+1)!/(1-a)^{k+2} \). So

\[
P_{n,a,f}(x) = \frac{1}{1-a} + \frac{(x-a)}{(1-a)^2} + \frac{(x-a)^2}{(1-a)^3} + \cdots + \frac{(x-a)^n}{(1-a)^{n+1}}.
\]

(b) Use the Lagrange form of the remainder term to show that if \( 0 < x < 1/2 \) then \( P_{n,0,f}(x) \to f(x) \) as \( n \to \infty \).

**Solution:** The Lagrange form of the remainder term gives

\[
R_{n,0,f}(x) = f^{(n+1)}(c) \frac{x^{n+1}}{(n+1)!} = \left( \frac{n+1}{1-c} \right) \frac{x^{n+1}}{(n+1)!} = \left( \frac{x}{1-c} \right)^{n+1}.
\]

where \( c \) is some number between 0 and \( x \). We have \( 0 < c < x < 1/2 \); but also we have \( x < 1/2 \); so \( 0 < x < 1-c \), and so \( 0 < \frac{x}{1-c} < 1 \). From this it seems reasonable that it might follow that \( \left( \frac{x}{1-c} \right)^{n+1} \to 0 \) as \( n \to \infty \), so \( R_{n,0,f}(x) \to 0 \), so \( P_{n,0,f}(x) \to f(x) \). But we have to be careful: \( c \) here is **not** a constant (necessarily); it changes as \( n \) changes (possible).
So it might be the case that while $\frac{x}{1-c} < 1$ for each $n$, we have $x/(1 - c)$ approaching arbitrarily close 1 as $n$ grows, making it difficult to conclude $\left(\frac{x}{1-c}\right)^{n+1} \to 0$; indeed, it might be that eventually

$$\frac{x}{1-c} = 1 - \frac{1}{n} (< 1),$$

in which case

$$\left(\frac{x}{1-c}\right)^{n+1} \to \frac{1}{e} \neq 0.$$ 

To rule something like this out, we have to realize that since $x$ is a fixed number below 1/2, we have $x = 1/2 - \varepsilon$ for some $\varepsilon > 0$, so from $1 - c > 1/2$ we get

$$\frac{x}{1-c} < 1 - 2\varepsilon;$$

and since $\varepsilon$ does not depend on $n$, we can now safely say

$$\left(\frac{x}{1-c}\right)^{n+1} < (1 - 2\varepsilon)^{n+1} \to 0.$$ 

The quiz as given out asked to show that $P_{n,a,f}(x) \to f(x)$ as $n \to \infty$ for $0 < x < 1/2$. But this is nonsense :(.