1. Give a complete statement of the ratio test for series convergence.

Solution: If \((a_n)\) is a sequence of non-negative terms, and \(\lim_{n \to \infty} \frac{a_{n+1}}{a_n}\) exists and is equal to \(r\) (acceptable: \(\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = r\)), then

- if \(r > 1\), \((a_n)\) is not summable (or, equally acceptable, \(\sum_{n=1}^{\infty} a_n\) does not converge);
- if \(r < 1\), \((a_n)\) is summable (or, equally acceptable, \(\sum_{n=1}^{\infty} a_n\) converges); and
- if \(r = 1\), not conclusion about the summability of \((a_n)\) (or, equally acceptable, about the convergence or otherwise of \(\sum_{n=1}^{\infty} a_n\)) can be reached.

Equally acceptable is a statement about the absolute summability of \((a_n)\) in terms of \(\lim_{n \to \infty} \frac{|a_{n+1}|}{|a_n|}\).

2. Determine for which real \(x\) the series \(\sum_{n=0}^{\infty} (-1)^n \frac{x^n}{2n+1}\) converges. Briefly justify your various assertions.

Solution: If \(x > 1\) or \(x < -1\) then the \(n\)th term does not go to zero, so the series does not converge for any of these values. If \(x = 1\) the series is

\[
1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots,
\]

which converges by the Leibniz alternating series test. If \(x = -1\) the series becomes

\[
1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \cdots,
\]

which does not converge, for example by the limit comparison test, comparing with the Harmonic series (the \(n\)th term of the Harmonic series is \(1/n\), and the \(n\) term of the present series is \(1/(2n - 1)\), so the limit of the ratio \(1/2\)). Finally, if \(-1 < x < 1\) we test for absolute convergence. The series with absolute values on all terms is \(\sum_{n=0}^{\infty} \frac{|x|^n}{2n+1}\), so \(a_n = |x|^n/(2n+1)\), and

\[
\frac{a_{n+1}}{a_n} = \frac{|x|^{n+1}(2n + 1)}{(2n + 3)|x|^n} = \frac{|x|(2n + 2)}{2n + 3} \to |x| < 1 \text{ as } n \to \infty,
\]

so by the ratio test, the series converges absolutely, so converges.

In summary, \(\sum_{n=0}^{\infty} (-1)^n \frac{x^n}{2n+1}\) converges for \(x \in (-1, 1]\) and nowhere else.