Math 10860, Honors Calculus 2

Final Exam

Solutions

1. (12 points)

   (a) (5 points) Use the definition of the Darboux integral to show that if $f : \mathbb{R} \to \mathbb{R}$ is an even function, and f is integrable, then for all $0 \leq a < b$ it holds that

$$\int_a^b f = \int_{-b}^{-a} f.$$

Solution: Let $P = \{t_0, t_1, \ldots, t_n\}$ (with $a = t_0 < t_1 < \cdots < t_n = b$) be any partition of $[a, b]$. Corresponding to this there is a partition P' of $[-b, -a]$, namely

$$P' = \{-t_n, -t_{n-1}, \ldots, -t_0\},$$

and every partition of $[-b, -a]$ arises from some partition of $[a, b]$ via this process (i.e., there is a 1-1 correspondence between partitions of $[a, b]$ and partitions of $[-b, -a]$, given by the process described above.)
Now for each i, and for each $x \in [t_{i-1}, t_i]$, we have $-x \in [-t_i, -t_{i-1}]$ and (by evenness of f) we have $f(-x) = f(x)$; conversely, for each $x \in [-t_i, -t_{i-1}]$ we have $-x \in [-t_i, -t_{i-1}]$, and again $f(-x) = f(x)$. It follows that for each i,

$$\{f(x) : x \in [t_{i-1}, t_i]\} = \{f(x) : x \in [-t_i, -t_{i-1}]\},$$

and so

$$\sup\{f(x) : x \in [t_{i-1}, t_i]\} = \sup\{f(x) : x \in [-t_i, -t_{i-1}]\}, \quad \inf\{f(x) : x \in [t_{i-1}, t_i]\} = \inf\{f(x) : x \in$$

From this we get that

$$U(f, P) = U(f, P') \quad \text{and} \quad L(f, P) = L(f, P').$$

Finally, the fact that for every P (for $[a, b]$) there is a corresponding P' (for $[-b, -a]$) says that

$$\inf U(f, P) = \inf U(f, P') \quad \text{and} \quad \sup L(f, P) = \sup L(f, P'),$$

so

$$\int_a^b f = \int_{-b}^{-a} f.$$

(b) (5 points) Suppose $h : [0,1] \to \mathbb{R}$ is a continuous function with $h(x) \geq 0$ for all $x \in [0,1]$, and with $\int_0^1 h(x)dx = 0$. Show that $h(x) = 0$ for all $x \in [0,1]$.

Solution: Suppose for a contradiction that there is some $x \in [0,1]$ with $h(x) > 0$. Then by continuity of h at x, there is some $\delta > 0$ such that $h(x') \geq h(x)/2$ for all $x \in [0,1]$ within δ of x.
If $x = 0$, consider the partition $P = (0, \delta, 1)$; it has lower Darboux sum at least $\delta h(x)/2$ (since for each $x' \in [0, \delta]$, we have $h(x') \geq h(x)/2$).
If $x = 1$, consider the partition $P = (0, 1-\delta, 1)$; it has lower Darboux sum at least $\delta h(x)/2$ (since for each $x' \in [1-\delta, \delta]$, we have $h(x') \geq h(x)/2$).
If $x \in (0,1)$, we can take δ small enough that $[x-\delta, x+\delta] \in [0,1]$. Consider the partition $P = (0, x-\delta, x+\delta, 1)$; it has lower Darboux sum at least $2\delta h(x)/2$ (since for each $x' \in [x-\delta, x+\delta]$, we have $h(x') \geq h(x)/2$).
So, whatever the value of x, there is a partition for which the lower Darboux sum is at least $\delta h(x)/2$, and hence

$$\int_0^1 h(x)dx \geq \delta h(x)/2 > 0,$$

the required contradiction.

(c) (2 points) Does the conclusion of the last part remain valid if the assumption of continuity is dropped?

Solution: No; consider for example the function that takes the value 1 at 1/2, and 0 everywhere else.

2. (10 points. If either of these are causing you a headache, ask me for a hint in exchange for a point.)

(a) (5 points) Let x be in $(0, \pi/2)$. Show that the expression

$$\int_{-\cos x}^{\sin x} \frac{dt}{\sqrt{1-t^2}}$$

does not depend on x, and find its value.

Solution: Letting $f(x) = \int_{-\cos x}^{\sin x} \frac{dt}{\sqrt{1-t^2}}$ we have

$$f(x) = \int_{-\cos x}^{0} \frac{dt}{\sqrt{1-t^2}} + \int_0^{\sin x} \frac{dt}{\sqrt{1-t^2}} = -\int_0^{-\cos x} \frac{dt}{\sqrt{1-t^2}} + \int_0^{\sin x} \frac{dt}{\sqrt{1-t^2}}.$$

Using FTOC (part 1) and chain rule, we get

$$\begin{aligned} f'(x) &= -\frac{\sin x}{\sqrt{1-(-\cos x)^2}} + \frac{\cos x}{\sqrt{1-(\sin x)^2}} \\ &= \frac{-\sin x}{\sin x} + \frac{\cos x}{\cos x} \\ &= 0, \end{aligned}$$

so that f is constant. Here we are using that on $(0, \pi/2)$ both sin and cos are strictly positive, so that $\sqrt{1-(\sin x)^2} = \cos x$ (as opposed to $-\cos x$) and $\sqrt{1-(-\cos x)^2} = \sin x$, and also so that we are not ever dividing by 0.

To compute the integral, we need only pick a single value of x, say $x \in \pi/2$, and use that the derivative of $\arcsin x$ is $1/\sqrt{1-x^2}$, to get

$$f(x) = f(\pi/2) = \int_{-\sqrt{2}/2}^{\sqrt{2}/2} \frac{dt}{\sqrt{1-t^2}} = [\arcsin x]_{x=-\sqrt{2}/2}^{\sqrt{2}/2} = \frac{\pi}{2}.$$

(b) (5 points) Compute $\lim_{x\to 0} \frac{1}{x^2} \int_0^x \frac{t}{1+t+e^t}\, dt$.

Solution: The fastest approach is via L'Hôpital's rule (since both numerator and denominator of

$$\frac{\int_0^x \frac{t}{1+t+e^t}\, dt}{x^2}$$

tend to 0 as x approaches 0). Using the fundamental theorem of calculus (part 1) for the numerator we have

$$\lim_{x\to 0} \frac{1}{x^2} \int_0^x \frac{t}{1+t+e^t}\, dt = \lim_{x\to 0} \frac{\frac{x}{1+x+e^x}}{2x} = \lim_{x\to 0} \frac{1}{2(1+x+e^x)} = \frac{1}{4}.$$

3. (14 points)

(a) (7 points) For $n \geq 0$, $n \in \mathbb{N}$, set

$$I_n = \int x^n \sqrt{1+x}\, dx.$$

i. (2 point) Find I_1 (i.e., as an expression that doesn't involve an integral).

Solution: Integration by parts, as used in the next part, is one possibility; here's another. Use the substitution $u = x+1$, so $du = dx$ and $x = (u-1)$ to get

$$\begin{aligned} I_1 &= \int x\sqrt{1+x}\, dx \\ &= \int (u-1)\sqrt{u}\, du \\ &= \int \left(u^{3/2} - u^{1/2}\right)\, du \\ &= \frac{2u^{5/2}}{5} - \frac{2u^{3/2}}{3} \\ &= \frac{2(x+1)^{5/2}}{5} - \frac{2(x+1)^{3/2}}{3}. \end{aligned}$$

ii. (5 points) Using integration by parts, find a reduction formula that expresses I_n in terms of I_{n-1}.

Solution: Take $u = x^n$ so $du = nx^{n-1}dx$ and $dv = \sqrt{1+x}$ so $v = \frac{2(1+x)^{3/2}}{3} = \frac{2}{3}(1+x)\sqrt{1+x}$ to get

$$\begin{aligned} I_n &= \int x^n \sqrt{1+x}\, dx \\ &= \frac{2x^n(1+x)^{3/2}}{3} - \frac{2n}{3}\int x^{n-1}(1+x)\sqrt{1+x}\, dx \\ &= \frac{2x^n(1+x)^{3/2}}{3} - \frac{2n}{3}\left(I_{n-1} + I_n\right). \end{aligned}$$

So

$$\left(1 + \frac{2n}{3}\right) I_n = \frac{2x^n(1+x)^{3/2}}{3} - \frac{2n}{3} I_{n-1}$$

or

$$I_n = \frac{2x^n(1+x)^{3/2}}{3+2n} - \frac{2n}{3+2n} I_{n-1}.$$

(b) (4 points) Find $\int \log\sqrt{1+x^2}\, dx$.

Solution: Use integration by parts, with

$$u = \log\sqrt{1+x^2}, \qquad du = \frac{2x\, dx}{2\sqrt{1+x^2}\sqrt{1+x^2}} = \frac{x}{1+x^2}$$

and

$$dv = dx, \qquad v = x,$$

to get

$$\begin{aligned} \int \log\sqrt{1+x^2}\, dx &= x\log\sqrt{1+x^2} - \int \frac{x^2\, dx}{1+x^2} \\ &= x\log\sqrt{1+x^2} - \int \left(1 - \frac{1}{1+x^2}\right) dx \\ &= x\log\sqrt{1+x^2} - x + \arctan(x). \end{aligned}$$

(c) (3 points) Does $\frac{3e^{3x}+2e^{2x}+e^x}{(1+e^x)(2+e^x)^2(3+e^x)^3}$ have an elementary primitive? (Briefly explain your reasoning, but please don't try to compute the integral — this should be short!)

Solution: Yes! Using substitution $u = e^x$ (so $du = e^x\, dx$ and $dx = du/e^x = du/u$) we get

$$\int \frac{3e^{3x}+2e^{2x}+e^x}{(1+e^x)(2+e^x)^2(3+e^x)^3}\, dx = \int \frac{3u^2+2u+1}{(1+u)(2+u)^2(3+u)^3}\, du.$$

This latter integral has an elementary primitive by partial fractions, and substituting back $u = e^x$ gives an elementary primitive for the original integrand.

4. (12 points)

(a) (4 points) Let $s_n(x) = 1 + x + x^2 + \cdots + x^n$ and let $f(x) = \frac{1}{1-x}$. Fix $x_0 \in (0,1)$. Carefully show that $s_n \to f$ *uniformly* on the interval $[-x_0, x_0]$[1].

Solution: For any $x \in [-x_0, x_0]$ we have

$$\begin{aligned} |f(x) - s_n(x)| &= \left| \frac{1}{1-x} - \frac{1-x^{n+1}}{1-x} \right| \\ &= \left| \frac{x^{n+1}}{1-x} \right| \\ &\leq \frac{|x|^{n+1}}{|1-x|}. \end{aligned}$$

Now we use

- $x \leq x_0$, so $1 - x \geq 1 - x_0 > 0$, so $0 < 1/(1-x) \leq 1/(1-x_0)$, so $1/|1-x| \leq 1/(1-x_0)$, and
- $|x| \leq |x_0|$, so $|x|^{n+1} \leq x_0^{n+1}$,

to get

$$|f(x) - s_n(x)| \leq \frac{x_0^{n+1}}{1-x_0}.$$

Because $0 < x_0 < 1$, given any ε there is an N (depending only on x_0, not on x) such that $n > N$ implies

$$\frac{x_0^{n+1}}{1-x_0} < \varepsilon$$

(since $x_0^{n+1}/(1-x_0) \to 0$ as $n \to \infty$). So for $n > N$, we have

$$|f(x) - s_n(x)| < \varepsilon.$$

This shows that $s_n \to f$ uniformly on $[-x_0, x_0]$.

(b) (8 points) Using what you know about the radius of convergence of the series $\sum_{n=0}^{\infty} x^n$ (and theorems about what happens inside the radius of convergence), evaluate each of the following sums, with brief justifications:

i. (4 points) $\sum_{n=0}^{\infty} (-1)^n \frac{n(n-1)}{3^n}$.

Solution: As shown in part (a), or from notes, or from the ratio test, the radius of convergence of $\sum_{n=0}^{\infty} x^n$ is 1. So on any interval of the form $[-x_0, x_0]$, $0 < x_0 < 1$, we have

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x},$$

and, differentiating term-by-term (valid inside the radius of convergence), we have

$$\sum_{n=0}^{\infty} n x^{n-1} = 1 + 2x + 3x^2 + \cdots = \frac{1}{(1-x)^2}.$$

[1] It may be helpful to remember that for $x \neq 1$, $1 + x + x^2 + \cdots + x^n = \frac{1-x^{n+1}}{1-x}$.

Differentiating term-by-term a second time (again, valid inside the radius of convergence), we have

$$\sum_{n=0}^{\infty} n(n-1)x^{n-2} = 2 + 6x + 12x^2 + \cdots = \frac{2}{(1-x)^3}.$$

At $x = -1/3$ (well inside the radius of convergence) we get

$$\sum_{n=0}^{\infty} (-1)^{n-2} \frac{n(n-1)}{3^{n-2}} = \frac{2}{(1+(1/3))^3} = \frac{27}{32},$$

so

$$\sum_{n=0}^{\infty} (-1)^n \frac{n(n-1)}{3^n} = \frac{1}{9} \sum_{n=0}^{\infty} (-1)^n \frac{n(n-1)}{3^{n-2}} = \frac{3}{32} = 0.09375.$$

ii. (4 points) $\sum_{n=1}^{\infty} \frac{2^n}{n3^n}$.

Solution: This time we integrate term-by-term (again, valid inside radius of convergence):

$$\begin{aligned} -\log(1-x) &= \int_0^x \frac{1}{1-t}\, dt \\ &= \int_0^x \left(\sum_{n=0}^{\infty} t^n \right) dt \\ &= \sum_{n=0}^{\infty} \int_0^x t^n \, dt \\ &= \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} \\ &= \sum_{n=1}^{\infty} \frac{x^n}{n}. \end{aligned}$$

Evaluating at $x = 2/3$ we get

$$\sum_{n=1}^{\infty} \frac{2^n}{n3^n} = -\log(1/3) = \log 3.$$

5. (6 points. This is the last question, of the last graded part, of first-year Honors Calculus. Make of that what you will...)

   (a) (2 points) Suppose that $f : (0,\infty) \to \mathbb{R}$ is twice-differentiable, and that there is an M_0 such that $|f(x)| \leq M_0$ and an M_2 such that $|f''(x)| \leq M_2$ for all $x \in (0,\infty)$. Prove that for all $h > 0$ and all $x \in (0,\infty)$,

$$|f'(x)| \leq \frac{2}{h} M_0 + \frac{h}{2} M_2.$$

Solution: For ant $x > 0$, apply Taylor's theorem (with the Lagrange form of the remainder term) to the Taylor polynomial (of degree 1) of f centered at x, to get that for all $h > 0$ we have

$$f(x+h) = f(x) + f'(x)h + f''(c)\frac{h^2}{2},$$

where c is some number between x and $x+h$. Dividing through by h and re-arranging, get

$$f'(x) = \frac{f(x+h)}{h} - \frac{f(x)}{h} - \frac{f''(c)h}{2}.$$

Taking absolute values of both sides and using the triangle inequality, get

$$|f'(x)| \leq \frac{|f(x+h)|}{h} + \frac{|f(x)|}{h} + \frac{|f''(c)|h}{2}.$$

By hypothesis $|f(x+h)|, |f(x)| \leq M_0$ and $|f''(c)| \leq M_2$, so

$$|f'(x)| \leq \frac{2}{h}M_0 + \frac{h}{2}M_2.$$

(b) (2 points) With the hypothesis as in part (a), deduce that for all $x \in (0, \infty)$,

$$|f'(x)| \leq 2\sqrt{M_0 M_2}.$$

Solution: The inequality proved in part (a) is valid for *any* $h > 0$. We would like to find the choice of $h > 0$ that makes the right-hand side of part (a) as small as possible.

Set $g(h) = (2/h)M_0 + (h/2)M_2$. We have $g'(h) = -(2/h^2)M_0 + M_2/2$. This derivative is negative on $(0, 2\sqrt{M_0/M_2})$ and positive on $(2\sqrt{M_0/M_2}, \infty)$, so on $(0, \infty)$ the function g has a global minimum at $2\sqrt{M_0/M_2}$. Applying the result of part (a) at this value, get

$$|f'(x)| \leq \frac{2}{2\sqrt{M_0/M_2}}M_0 + \frac{2\sqrt{M_0/M_2}}{2}M_2 = 2\sqrt{M_0 M_2}.$$

(c) (2 points) Suppose that $f : (0, \infty) \to \mathbb{R}$ is twice-differentiable, that $f''(x)$ is bounded, and that $f(x) \to 0$ as $x \to \infty$. Prove that $f'(x) \to 0$ as $x \to \infty$.

Solution: The results of parts (a) and (b) go through without any change, when applied to the interval (a, ∞) for any $a \geq 0$. In other words:

> Suppose that $f : (a, \infty) \to \mathbb{R}$ is twice-differentiable, and that there is an $M_{0,a}$ such that $|f(x)| \leq M_{0,a}$ and an $M_{2,a}$ such that $|f''(x)| \leq M_{2,a}$ for all $x \in (a, \infty)$. Then for all $x \in (a, \infty)$,
>
> $$|f'(x)| \leq 2\sqrt{M_{0,a} M_{2,a}}.$$

For the given function f of the problem, we may take $M_{2,0} = M$ for some M that doesn't depend on a (we are given that $f''(x)$ is bounded). So we can say that for all $x \in (a, \infty)$,

$$|f'(x)| \leq 2\sqrt{M_{0,a}M}$$

for some absolute constant M.

Now, given that $f(x) \to 0$ as $x \to \infty$, given $\varepsilon > 0$ there is a such that on (a, ∞) we have $|f(x)| < \frac{\varepsilon^2}{4M}$, so on (a, ∞) we may take $M_{0,a} = \frac{\varepsilon^2}{4M}$, and we get that for $x > a$,

$$|f'(x)| \leq 2\sqrt{M}\sqrt{\frac{\varepsilon^2}{4M}} = \varepsilon.$$

Since ε was arbitrary, this shows that $|f'(x)| \to 0$ as $x \to \infty$.