Math 10860, Honors Calculus 2

Homework 10 **NAME**:

Due by 11pm Tuesday April 28

Instructions

Yadda yadda yadda. Email to `math10860homework at gmail.com`.

Reading

Class notes, Sections 15.4 (on Bolzano-Weierstrass) and Chapter 16 (on series).

Assignment

1. Use the Bolzano-Weierstrass theorem (every bounded sequence has a convegent subsequence) to prove the first part of the Extreme Value Theorem: if $f : [a, b] \to \mathbb{R}$ is continuous, then there is M such that $f(x) \leq M$ for all $x \in [a, b]$. (**Hint**: Try a proof by contradiction.)

2. Decide whether the following sums converge. Explain your reasoning (i.e., which tests you are using, and why they apply.)

   - $\sum_{n=1}^{\infty}(-1)^n \frac{\log n}{n}$.
   - $\sum_{n=1}^{\infty} \frac{n^2}{n!}$.
   - $\sum_{n=1}^{\infty} \frac{1}{(\log n)^n}$.
   - $\sum_{n=2}^{\infty} \frac{1}{n(\log n)^2}$.
   - $\sum_{n=1}^{\infty} \frac{1}{n^{1+1/n}}$.

3. (a) In the sum below, a is positive. Use the ratio test to decide for which values of a the sum converges, and for which values it diverges:

   $$\sum_{n=1}^{\infty} \frac{a^n n!}{n^n}.$$

   (b) You should find that the ratio test gives no information at $a = e$ (if you didn't: redo part (a)!). When $a = e$, show that the series diverges, by using a result from the last homework.

(c) Decide when

$$\sum_{n=1}^{\infty} \frac{n^n}{a^n n!}$$

converges, again using a result from the last homework when the ratio test fails.

4. Leibniz' alternating series test says that if (a_n) is a non-increasing sequence of non-negative numbers, and $(a_n) \to 0$ as $n \to 0$, then $\sum_{n=1}^{\infty} (-1)^n/n$ is finite.

   Is the hypothesis "non-increasing" necessary, or is the conclusion still valid if we merely assume that non-negative a_n tends to 0?

5. (a) Prove that if $a_n \geq 0$ and (a_n) is not summable (i.e., $\sum a_n$ diverges), then $(a_n/(1+a_n))$ is not summable.

   (b) Is the converse true? If $(a_n/(1+a_n))$ is not summable (with $a_n > 0$), must it always be the case that (a_n) is not summable?

6. Wally, a slow but persistent worm, starts at one end of a meter-long rubber band and crawls one centimeter per minute toward the other end. At the end of each minute Wally rests for a moment. During that moment Karl, the equally persistent keeper of the band (whose sole purpose in life is to frustrate Wally) stretches the band one meter. Thus after one minute of crawling, Wally is 1 centimeter from the start and 99 from the finish; but in the moment that Wally rests, Karl stretches the band one meter. During the stretching Wally maintains his relative position, 1% from the start and 99% from the finish. So at the end of Wally's moment of rest, he is 2cm from the starting point and 198cm from his goal. After Wally crawls for another minute the score is 3cm traveled and 197 to go; but then Karl stretches one more meter (from 2 meters to 3 meters), and Wally's distances become 4.5cm travelled, and 295.5cm to go. And so on.

   Does Wally ever get to the end of the band???

   He keeps moving, but the goal seems to be moving away from him, faster than he moves. (We're assuming here infinite longevity for Karl and Wally, infinite elasticity of the band, and an infinitely tiny worm.)

7. Define the 7-*depleted harmonic number* $H_n^{(7)}$ to be the sum of the reciprocals of the natural numbers from 1 to n, *except* those n that have a 7 in their decimal expansion. For example, $H_8^{(7)} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{8}$. (There is no standard name or notation for this number).

   Does $(H_n^{(7)})_{n=1}^{\infty}$ converge or diverge?

8. (a) Suppose that $(a_n)_{n=1}^{\infty}$ is weakly decreasing, with $a_n \geq 0$, and that $\sum_{n=1}^{\infty} a_n$ is finite. The vanishing condition says that $\lim_{n\to\infty} a_n = 0$. Prove something stronger: $\lim_{n\to\infty} n a_n = 0$.

   (b) For each $\alpha > 0$, give an example of a sequence $(a_n)_{n=1}^{\infty}$ that is weakly decreasing, with $a_n \geq 0$, with $\sum_{n=1}^{\infty} a_n$ is finite, but with $\lim_{n\to\infty} n^{1+\alpha} a_n = +\infty$ (so, the result you proved in part (a) can't be improved upon).

   (c) Is the hypothesis "weakly decreasing" necessary?