Math 10860, Honors Calculus 2

Homework 10 **NAME**:

Solutions

1. Use the Bolzano-Weierstrass theorem (every bounded sequence has a convegent subsequence) to prove the first part of the Extreme Value Theorem: if $f : [a, b] \to \mathbb{R}$ is continuous, then there is M such that $f(x) \leq M$ for all $x \in [a, b]$. (**Hint**: Try a proof by contradiction.)

   Solution: Suppose there is no such M. Then for each $n \in \mathbb{N}$ there is $x_n \in [a, b]$ with $f(x_n) > n$.

   The sequence $(x_n)_{n\geq 1}$ is bounded (by a and b), so has (by Bolzano-Weierstrass) a convergent subsequence $(a_{n_i})_{i\geq 1}$. This subsequence converges to a limit, say ℓ, that is between a and b (if the limit was greater than b, then eventually the terms of the subsequence would be greater than b, not possible; ditto if the limit was less than a).

   f is continuous at ℓ (by hypothesis), so by the limit of sequences/continuity theorem, since $x_{n_i} \to \ell$ we have $f(x_{n_i}) \to f(\ell)$. But clearly $f(x_{n_i}) \to +\infty$, a contradiction.

   Hence there *is* an M with $f(x) \leq M$ for all $x \in [a, b]$.

2. Decide whether the following sums converge. Explain your reasoning (i.e., which tests you are using, and why they apply.)

   - $\sum_{n=1}^{\infty} (-1)^n \frac{\log n}{n}$.

     Solution: Converges; eventually (after $n = 3$) the nth term (in absolute value) is decreasing, and tends to 0, so Leibniz' test works after finitely many terms at the beginning are thrown out.

   - $\sum_{n=1}^{\infty} \frac{n^2}{n!}$.

     Solution: Converges (in a hurry!) by ratio test.

   - $\sum_{n=1}^{\infty} \frac{1}{(\log n)^n}$.

     Solution: Converges; ratio test should work here.

   - $\sum_{n=2}^{\infty} \frac{1}{n(\log n)^2}$.

     Solution: Converges; integral test certainly works.

- $\sum_{n=1}^{\infty} \frac{1}{n^{1+1/n}}$.

  Solution: Let $a_n = 1/(n^{1+1/n})$ and let $b_n = 1/n$ (note $\sum_{n=1}^{\infty} b_n$ diverges). We have
$$\frac{a_n}{b_n} = \frac{n}{n^{1+1/n}} = \frac{1}{n^{1/n}} \to 1 \quad \text{as } n \to \infty$$
  (since $\lim_{n\to\infty} n^{1/n} = \lim_{n\to\infty} e^{(\log n)/n} = e^0 = 1$), and so, by the limit comparison test $\sum_{n=1}^{\infty} a_n$ diverges.

3. (a) In the sum below, a is positive. Use the ratio test to decide for which values of a the sum converges, and for which values it diverges:
$$\sum_{n=1}^{\infty} \frac{a^n n!}{n^n}.$$

   Solution: We use the ratio test. Let $a_n = (a^n n!)/(n^n)$. We have
$$\frac{a_{n+1}}{a_n} = \frac{a^{n+1}(n+1)!n^n}{(n+1)^{n+1}a^n n!} = \frac{a}{\left(1+\frac{1}{n}\right)^n} \to \frac{a}{e} \quad \text{as } n \to \infty$$
   (using $(1 + 1/n)^n \to e$ as $n \to \infty$). So if $0 < a < e$ the series converges ($\lim_{n\to\infty} a_{n+1}/a_n < 1$) and if $a > e$ it diverges.

   (b) You should find that the ratio test gives no information at $a = e$ (if you didn't: redo part (a)!). When $a = e$, show that the series diverges, by using a result from the last homework.

   Solution: We know (Homework 9, question 6, part (c))
$$\lim_{n\to\infty} \frac{\sqrt[n]{n!}}{n} = \frac{1}{e}.$$
   Fix $\varepsilon > 0$. There is $n_0 = n_0(\varepsilon)$ such that for $n \geq n_0$ we have
$$\frac{e\sqrt[n]{n!}}{n} \geq 1 - \varepsilon,$$
   so
$$\frac{e^n n!}{n^n} \geq (1-\varepsilon)^n.$$
   It follows that
$$\sum_{n=1}^{\infty} \frac{e^n n!}{n^n} \geq \sum_{n=n_0}^{\infty} \frac{e^n n!}{n^n} \geq \sum_{n=n_0}^{\infty} (1-\varepsilon)^n = \frac{(1-\varepsilon)^{n_0}}{1-(1-\varepsilon)} = \frac{(1-\varepsilon)^{n_0}}{\varepsilon}.$$
   It's tempting to say that because of the ε in the denominator, as ε gets smaller $(1-\varepsilon)^{n_0}/\varepsilon$ gets larger (goes to infinity) so the sum is bounded; unfortunately we can't do this, since n_0 depends on ε too, and the $(1-\varepsilon)^{n_0}$ may also be getting very small as ε gets small.

Instead we need to re-use Homework 9, question 6, this time part (b). We have

$$n! > \frac{n^n}{e^{n-1}}$$

and so

$$\frac{e^n n!}{n^n} > \frac{1}{e}.$$

By the vanishing condition it immediately follows that $\sum_{n=1}^{\infty} \frac{e^n n!}{n^n}$ diverges.

(c) Decide when

$$\sum_{n=1}^{\infty} \frac{n^n}{a^n n!}$$

converges, again using a result from the last homework when the ratio test fails.

Solution: The series converges for $a > e$ by the ratio test, and diverges for $a < e$. At $a = e$ an analysis very similar to the last part gives that the series diverges.

4. Leibniz' alternating series test says that if (a_n) is a non-increasing sequence of non-negative numbers, and $(a_n) \to 0$ as $n \to 0$, then $\sum_{n=1}^{\infty}(-1)^n/n$ is finite.

Is the hypothesis "non-increasing" necessary, or is the conclusion still valid if we merely assume that non-negative a_n tends to 0?

Solution: The hypothesis is necessary. Consider, e.g., the sequence given by

$$a_n = \begin{cases} \frac{1}{m} & \text{if } n \text{ odd, say } n = 2m+1,\ m = 0, 1, 2, 3, \dots \\ 0 & \text{if } n \text{ even.} \end{cases}$$

Here (a_n) is a sequence of non-negative numbers, and $(a_n) \to 0$ as $n \to 0$, but $\sum_{n=1}^{\infty}(-1)^n/n$ diverges (it is the harmonic sum).

5. (a) Prove that if $a_n \geq 0$ and (a_n) is not summable (i.e., $\sum a_n$ diverges), then $(a_n/(1+a_n))$ is not summable.

Solution: We prove the contrapositive. Suppose $(a_n/(1+a_n))$ is summable. We have that $a_n/(1+a_n) \to 0$ as $n \to \infty$ (a necessary condition for summability), so for $\varepsilon > 0$ there's n_0 such that $n > n_0$ implies

$$\frac{a_n}{1+a_n} < \varepsilon.$$

This is equivalent to

$$a_n < \frac{\varepsilon}{1-\varepsilon}.$$

Note that as ε approaches 0 from above, so does $\varepsilon/(1-\varepsilon)$, so we conclude that $a_n \to 0$ as $n \to \infty$.

Now compare (a_n) and $(a_n/(1+a_n))$. We have

$$\frac{a_n}{a_n/(1+a_n)} = 1 + a_n \to 1 \quad \text{as } n \to \infty,$$

so by (limit) comparison (a_n) is summable.

(b) Is the converse true? If $(a_n/(1+a_n))$ is not summable (with $a_n > 0$), must it always be the case that (a_n) is not summable?

Yes. Again we prove the contrapositive, that if (a_n) is summable then so is $(a_n/(1+a_n))$. This is a direct proof by limit comparison:

$$\frac{a_n}{a_n/(1+a_n)} = 1 + a_n \to 1 \quad \text{as } n \to \infty$$

since, by summability of (a_n), $a_n \to 0$ as $n \to \infty$.

6. Wally, a slow but persistent worm, starts at one end of a meter-long rubber band and crawls one centimeter per minute toward the other end. At the end of each minute Wally rests for a moment. During that moment Karl, the equally persistent keeper of the band (whose sole purpose in life is to frustrate Wally) stretches the band one meter. Thus after one minute of crawling, Wally is 1 centimeter from the start and 99 from the finish; but in the moment that Wally rests, Karl stretches the band one meter. During the stretching Wally maintains his relative position, 1% from the start and 99% from the finish. So at the end of Wally's moment of rest, he is 2cm from the starting point and 198cm from his goal. After Wally crawls for another minute the score is 3cm traveled and 197 to go; but then Karl stretches one more meter (from 2 meters to 3 meters), and Wally's distances become 4.5cm travelled, and 295.5cm to go. And so on.

Does Wally ever get to the end of the band???

He keeps moving, but the goal seems to be moving away from him, faster than he moves. (We're assuming here infinite longevity for Karl and Wally, infinite elasticity of the band, and an infinitely tiny worm.)

Solution: YES!

In one time unit, Wally traverses 1/100 of the length of the band. During the stretching, he doesn't change his *relative* progress: he has still covered 1/100 of the band. In the next time unit, he covers an addition proportion 1/200 of the band (1cm out of 2m), so he has now covered a total proportion of $1/100 + 1/200$ of the band. Again, this doesn't change during stretching. After the next time unit (in which he covers an additional 1cm out of 3m, 1/300 of the band) he has covered a total proportion $1/100 + 1/200 + 1/300$. In general, after n time units, he has covered a proportion

$$\frac{1}{100} + \frac{1}{200} + \cdots + \frac{1}{100n} = \frac{H_n}{100}$$

of the band. Because $(H_n) \to \infty$, there is an n for which $H_n \geq 100$ for the first time; at this point Wally has completed his traversal of the band.

Note that $n \approx 2.6 \times 10^{43}$(!)

7. Define the 7-*depleted harmonic number* $H_n^{(7)}$ to be the sum of the reciprocals of the natural numbers from 1 to n, *except* those n that have a 7 in their usual decimal

representation. For example, $H_8^{(7)} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{8}$. (There is no standard name or notation for this number).

Does $(H_n^{(7)})_{n=1}^{\infty}$ converge or diverge?

Solution: The sum, surprisingly, converges. It is sometimes referred to as the *Kempner sum*. See `https://en.wikipedia.org/wiki/Kempner_series` for a discussion.

Here's a proof of the divergence:

- There are 8 one-digit natural numbers that have no 7 in their base-10 representation. Each of these contributes at most 1 to the sum, so the total contribution from one-digit numbers is at most 8.
- There are 8×9 two-digit natural numbers that have no 7 in their base-10 representation — 8 options for the first digit (1, 2, 3, 4, 5, 6, 8 or 9) and 9 options for the second (0, 1, 2, 3, 4, 5, 6, 8 or 9). Each of these contributes at most $1/10$ to the sum (10 is the smallest two-digit number), so the total contribution from two-digit numbers is at most $8 \times 9/10$
- In general there are $8 \times 9^{n-1}$ n-digit natural numbers that have no 7 in their base-10 representation — 8 options for the first digit and 9 options for each of the remaining $n-1$ digits. Each of these contributes at most $1/10^{n-1}$ to the sum, so the total contribution from two-digit numbers is at most $8 \times 9^{n-1}/10^{n-1}$.

It follows that $s_{9\ldots9}$, where there are n 9's, is at most

$$8 + 8 \times \frac{9}{10} + 8\left(\frac{9}{10}\right)^2 + \cdots + 8\left(\frac{9}{10}\right)^{n-1}.$$

This partial sum is bounded above by

$$\sum_{n=0}^{\infty} 8\left(\frac{9}{10}\right)^n = \frac{8}{1-\frac{9}{10}} = 80,$$

independent of n.

So the partial sums are increasing and bounded above (by 80), and the 7-depleted Harmonic sum converges (and has sum at most 80).

Here's the intuition: most large numbers have all 10 digits in them, so most natural numbers are being stripped from the Harmonic sum to get the depleted Harmonic sum.

More generally, if A is any natural number, then we can define the *A-depleted harmonic number* $H_n^{(A)}$ to be the sum of the reciprocals of the natural numbers from 1 to n, *except* those n that have the string A appearing consecutively in their usual decimal representation. Bu essentially the same proof, done more carefully, we get that the sequence (H_n^A) is summable!

8. (a) Suppose that $(a_n)_{n=1}^{\infty}$ is weakly decreasing, with $a_n \geq 0$, and that $\sum_{n=1}^{\infty} a_n$ is finite. The vanishing condition says that $\lim_{n\to\infty} a_n = 0$. Prove something stronger: $\lim_{n\to\infty} na_n = 0$.

Solution: Given $\varepsilon > 0$, there is N such that $n > m \geq N$ implies that

$$|s_n - s_N| = a_n + a_{n-1} + \cdots + a_m < \varepsilon/2$$

(the Cauchy criterion for convergence). Apply this with $m = N$ to get

$$a_n + a_{n-1} + \cdots + a_{N+1} < \varepsilon/2$$

Now using that the sequence is non-decreasing, we get

$$a_n + a_{n-1} + \cdots + a_{N+1} \geq (n - N)a_n = na_n - Na_n,$$

so that for all $n > N$,

$$na_n - Na_n < \varepsilon/2 \quad \text{or} \quad na_n < \varepsilon/2 + Na_n.$$

Now $a_n \to 0$ by the vanishing criterion, so there is $M > 0$ such that $n \geq M$ implies $a_n < \varepsilon/(2N)$. So for $n \geq \max\{N, M\}$, have

$$na_n < \varepsilon/2 + Na_n < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, we have shown that $na_n \to 0$.

(b) For each $\alpha > 0$, give an example of a sequence $(a_n)_{n=1}^{\infty}$ that is weakly decreasing, with $a_n \geq 0$, with $\sum_{n=1}^{\infty} a_n$ is finite, but with $\lim_{n\to\infty} n^{1+\alpha} a_n = +\infty$ (so, the result you proved in part (a) can't be improved upon).

Solution: Take $a_n = 1/n^{1+\alpha/2}$. (It is non-negative, weakly decreasing to 0, and summable by the integral test; but $\lim_{n\to\infty} n^{1+\alpha} a_n = \lim_{n\to\infty} n^{\alpha/2} = +\infty$.)

(c) Is the hypothesis "weakly decreasing" necessary?

Solution: Yes. For example, consider the sequence given by

$$a_n = \begin{cases} \frac{1}{n} & \text{if } n = 1, 4, 9, 16, 25, 36, \ldots \\ 0 & \text{otherwise.} \end{cases}$$

Here $a_n \geq 0$, and $\sum_{n=1}^{\infty} a_n = 1 + 1/4 + 1/9 + \cdots$ is finite. But $\lim_{n\to\infty} na_n$ does not exist: for all n that is not a perfect square, $na_n = 0$, while for all n that is a perfect square, $na_n = 1$.