Math 10860, Honors Calculus 2

Homework 5 **NAME**:

Solutions

Reading for this homework

Sections 12.1 and 12.2 of the course notes.

Solutions

1. Differentiate these functions: (Convention: a^{b^c} always means $a^{(b^c)}$.)

   (a)
$$f(x) = e^{e^{e^{e^x}}}.$$

   Solution:
$$f'(x) = e^{e^{e^{e^x}}} \cdot e^{e^{e^x}} \cdot e^{e^x} \cdot e^x.$$

   (b)
$$f(x) = e^{\left(\int_0^x e^{-t^2}\,dt\right)}.$$

   Solution:
$$f'(x) = e^{\left(\int_0^x e^{-t^2}\,dt\right)} \cdot e^{-x^2}.$$

   (c)
$$f(x) = (\log x)^{\log x}.$$

   Solution: $f(x) = e^{\left(\log\left((\log x)^{\log x}\right)\right)} = e^{\log x \log\log x}$, so
$$\begin{aligned} f'(x) &= e^{\log x \log\log x} \cdot \left(\log x \cdot \frac{1}{x\log x} + \frac{\log\log x}{x}\right) \\ &= (\log x)^{\log x} \cdot \left(\frac{1+\log\log x}{x}\right). \end{aligned}$$

2. The *logarithmic derivative* of f is the expression f'/f. It's called "logarithmic derivative" because it is the derivative of $\log\circ f$. It is often easier to compute the derivative of log of a function than it is to compute the derivative of the function directly, because taking logs turns products into (simpler to differentiate) sums, and turns powers into (simpler to differentiate) products. The derivate of the original function can then be recovered by multiplying by the original function.

   Compute the logarithmic derivatives of these functions:

(a)

$$f(x) = x^x.$$

Solution: $\log f(x) = x \log x$, so

$$(\log \circ f)'(x) = \frac{f'(x)}{f(x)} = x\frac{1}{x} + \log x = 1 + \log x.$$

(b)

$$f(x) = \frac{(3-x)^{1/3}x^2}{(1-x)(3+x)^{2/3}}.$$

Solution:

$$\log f(x) = \frac{\log(3-x)}{3} + 2\log x - \log(1-x) - \frac{2\log(3+x)}{3}$$

so

$$\frac{f'(x)}{f(x)} = \frac{-1}{3(3-x)} + \frac{2}{x} + \frac{1}{1-x} - \frac{2}{3(3+x)}.$$

(c)

$$f(x) = \frac{e^x - e^{-x}}{e^{2x}(1+x^3)}.$$

Solution:

$$\log f(x) = \log(e^x - e^{-x}) - 2x - \log(1+x^3)$$

so

$$\frac{f'(x)}{f(x)} = \frac{e^x + e^{-x}}{e^x - e^{-x}} - 2 - \frac{3x^2}{1+x^3}.$$

3. Compute these limits:

(a)

$$\lim_{x\to 0} \frac{e^x - 1 - x - x^2/2}{x^2}.$$

Solution: We have, by a direct evaluation,

$$\lim_{x\to 0} \frac{e^x - 1}{2} = 0$$

and so, via L'Hôpital's rule,

$$\lim_{x\to 0} \frac{e^x - 1 - x}{2x} = 0$$

and so, again by L'Hôpital's rule,

$$\lim_{x\to 0} \frac{e^x - 1 - x - x^2/2}{x^2} = 0.$$

(b)

$$\lim_{x\to\infty} \frac{x}{(\log x)^n} \qquad (n \text{ a natural number}).$$

Solution: This is an indeterminate of the form ∞/∞, so we apply L'Hôpital's rule, to get

$$\lim_{x\to\infty} \frac{x}{(\log x)^n} = \lim_{x\to\infty} \frac{1}{(n(\log x)^{n-1})/x} = \lim_{x\to\infty} \frac{x}{n(\log x)^{n-1}}$$

$n-1$ more applications of L'Hôpital's rule lead to

$$\lim_{x\to\infty} \frac{x}{(\log x)^n} = \lim_{x\to\infty} \frac{x}{n!\log x}$$

and one more application of L'Hôpital's rule leads to

$$\lim_{x\to\infty} \frac{x}{(\log x)^n} = \lim_{x\to\infty} \frac{x}{n!} = \infty.$$

(c)

$$\lim_{x\to 0^+} \frac{x}{(\log x)^n} \qquad (n \text{ a natural number}).$$

Solution: Since $\log x \to -\infty$ as $x \to 0^+$, we have that $1/\log x \to 0$ as $x \to 0^+$, and so for any natural number n, it follows that $(1/\log x)^n = 1/(\log x)^n \to 0$ as $x \to 0^+$, and so $x/(\log x)^n \to 0$ as $x \to 0^+$.

(d)

$$\lim_{x\to 0^+} x^x.$$

Solution: We have $x^x = e^{x\log x}$, so we should examine what happens to $x \log x$ as x approaches 0 from above. We have

$$\lim_{x\to 0^+} x\log x = \lim_{x\to 0^+} \frac{\log x}{1/x} = \lim_{x\to 0^+} \frac{1/x}{-1/x^2} = \lim_{x\to 0^+} -x = 0,$$

with the first equality a bit of algebraic manipulation, and the second an application of L'Hôpital's rule. It follows (after a short ε-δ argument that's omitted here, but appears elsewhere in these solutions) that

$$\lim_{x\to 0^+} e^{x\log x} = e^0 = 1,$$

so $\lim_{x\to 0^+} x^x = 1$.

4. Which number is bigger: e^π or π^e? (Rigorously justify your answer!)

Solution: Consider the function $f : (0,\infty) \to \mathbb{R}$ defined by

$$f(x) = x^{\frac{1}{x}} = e^{\frac{\log x}{x}}.$$

We have

$$f'(x) = e^{\frac{\log x}{x}}\left(\frac{1-\log x}{x^2}\right).$$

This is positive on $(0, e)$ and negative on (e, ∞), so f has a global maximum at $x = e$. It follows that for any number $x \neq e$, we have $f(e) > f(x)$, or

$$e^{\frac{1}{e}} > x^{\frac{1}{x}}.$$

Raising both sides to the power ex yields

$$e^x > x^e.$$

In particular, since $e \neq \pi$ we have

$$e^\pi > \pi^e.$$

5. Prove that $F(x) = \int_2^x \frac{dt}{\log t}$ is not a bounded function on $[2, \infty)$.

   Solution: We claim that there is t_0 such that for $t \geq t_0$ we have $\log t \leq t$, so that $1/\log t \geq 1/t$. That implies (from the comparison theorem that we proved in class) that

$$\begin{aligned} \lim_{x\to\infty}\int_2^x \frac{dt}{\log t} &= \int_2^{t_0}\frac{dt}{\log t} + \lim_{x\to\infty}\int_{t_0}^x \frac{dt}{\log t} \\ &\geq \int_2^{t_0}\frac{dt}{\log t} + \lim_{x\to\infty}\int_{t_0}^x \frac{dt}{t} \\ &= \infty. \end{aligned}$$

   To see the claim, note that since $\lim_{t\to\infty} e^t/t = \infty$ there is a t_0 such that for $t \geq t_0$ we have $t \leq e^t$, and so, taking logarithms (valid since log is an increasing function), $\log t \leq t$.

   Meta-question: Why am I asking this question? There is an important mathematical concept, one that you've been familiar with for many years, and one that most non-mathematics are familiar with, that this integral is intimately related to. What is the concept, and what is the connection?

   Solution: For a positive number x, let $\pi(x)$ denote the number of prime numbers less than or equal to x (so, for example, $\pi(9) = \pi(9.2) = 4$ and $\pi(29) = 10$). One version of the *Prime number theorem*, a central result in number theory, asserts that the function $F(x)$ above gives a very good approximation to $\pi(x)$ for large x, in the sense that

$$\lim_{x\to\infty}\frac{\pi(x)}{F(x)} = 1.$$

6. This question guides you to an alternate expression for e.

(a) Find $\lim_{y \to 0} \frac{\log(1+y)}{y}$.

Solution: By L'Hôpital's rule

$$\lim_{y \to 0} \frac{\log(1+y)}{y} = \lim_{y \to 0} \frac{\frac{1}{1+y}}{1} = 1.$$

(b) Find $\lim_{x \to \infty} x \log(1 + 1/x)$.

Solution: We know that in general $\lim_{x \to \infty} f(x) = \lim_{y \to 0^+} f(1/y)$ (we proved this last semester). So

$$\lim_{x \to \infty} x \log(1 + 1/x) = \lim_{y \to 0^+} \frac{\log(1+y)}{y} = 1.$$

(c) Prove that

$$e = \lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x.$$

Solution: Fix $\varepsilon > 0$.

Because $e^0 = 1$, and exp is continuous at 0, we know that there is a $\delta > 0$ such that $|x - 0| < \delta$ implies $|e^x - 1| < \varepsilon$. In other words, for $-\delta < x < \delta$,

$$1 - \varepsilon < e^x < 1 + \varepsilon \quad (\star).$$

We also know that $\lim_{x \to \infty} x \log(1 + 1/x) = 1$, so there is N such that $x > N$ implies

$$|x \log(1 + 1/x) - 1| < \delta/2.$$

So, for $x > N$ we have

$$-\frac{\delta}{2} < x \log(1 + 1/x) - 1 < \frac{\delta}{2}.$$

Exponentiating both sides, and using that exp is increasing, we get

$$e^{-\delta/2} < \frac{\left(1 + \frac{1}{x}\right)^x}{e} < e^{\delta/2}.$$

But now we can apply $(\star)$ (since $\delta/2$ and $-\delta/2$ are both between $-\delta$ and δ) to get

$$1 - \varepsilon < e^{-\delta/2} < \frac{\left(1 + \frac{1}{x}\right)^x}{e} < e^{\delta/2} < 1 + \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, this shows that

$$\lim_{x \to \infty} \frac{\left(1 + \frac{1}{x}\right)^x}{e} = 1,$$

which implies that

$$\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x = e.$$

(d) Go though the same process to argue that for all real a

$$e^a = \lim_{x\to\infty}\left(1+\frac{a}{x}\right)^x.$$

Specifically:

- First, argue $\lim_{y\to 0}\frac{\log(1+ay)}{y} = a$.
- Next, argue $\lim_{x\to\infty} x\log(1+a/x) = a$.
- Finally, argue $e^a = \lim_{x\to\infty}\left(1+\frac{a}{x}\right)^x$.

Solution:
Step 1: ($\lim_{y\to 0}\frac{\log(1+ay)}{y} = a$)
By L'Hôpital's rule

$$\lim_{y\to 0}\frac{\log(1+ay)}{y} = \lim_{y\to 0}\frac{\frac{a}{1+ay}}{1} = a.$$

Step 2: ($\lim_{x\to\infty} x\log(1+a/x) = a$)
We know that in general $\lim_{x\to\infty} f(x) = \lim_{y\to 0^+} f(1/y)$. So

$$\lim_{x\to\infty} x\log(1+a/x) = \lim_{y\to 0^+}\frac{\log(1+ay)}{y} = a.$$

Step 3: ($e^a = \lim_{x\to\infty}\left(1+\frac{a}{x}\right)^x$)
Fix $\varepsilon > 0$. We repeat the same argument as before.
Because exp is continuous at 0, we know that there is a $\delta > 0$ such that for $-\delta < x - a < \delta$,

$$1-\varepsilon < e^x < 1+\varepsilon \quad (\star).$$

We also know that $\lim_{x\to\infty} x\log(1+a/x) = a$, so there is N such that $x > N$ implies

$$|x\log(1+a/x) - a| < \delta/2.$$

So, for $x > N$ we have

$$-\frac{\delta}{2} < x\log(1+a/x) - a < \frac{\delta}{2}.$$

Exponentiating both sides, and using that exp is increasing, we get

$$e^{-\delta/2} < \frac{\left(1+\frac{a}{x}\right)^x}{e^a} < e^{\delta/2}.$$

But now we can apply ($\star$) (since $\delta/2$ and $-\delta/2$ are both between $-\delta$ and δ) to get

$$1-\varepsilon < e^{-\delta/2} < \frac{\left(1+\frac{a}{x}\right)^x}{e^a} < e^{\delta/2} < 1+\varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, this shows that

$$\lim_{x\to\infty}\frac{\left(1+\frac{a}{x}\right)^x}{e^a} = 1,$$

which implies that

$$\lim_{x\to\infty}\left(1+\frac{a}{x}\right)^x = e^a.$$

7. This gives an alternate proof of a basic estimate we proved in class.

   (a) Prove that for all natural numbers $n \geq 1$ and for all real $x > 0$ we have

$$\sum_{k=0}^{n} \frac{x^k}{k!} \leq e^x.$$

   Solution: We'll proceed by induction on n The base case $n = 1$ is the assertion that $e^x \geq 1 + x$ for $x > 0$. We'll actually prove the inequality for $x \geq 0$, using the following lemma we proved in class:

   Lemma: if $f, g : [a, \infty) \to \mathbb{R}$ are both differentiable, satisfy $f(a) = g(a)$ and $f' \geq g'$ on $[a, \infty)$, then $f \geq g$ on $[a, \infty)$.

   The base case follows from this taking $f(x) = e^x$, $g(x) = 1 + x$ and $a = 0$.

   We now move to the induction step. Suppose that for some $n \in \mathbb{N}$, $\sum_{k=0}^{n} \frac{x^k}{k!} \leq e^x$ for all $x > 0$. We wish to show that

$$e^x \geq 1 + x + \frac{x^2}{2} + \cdots + \frac{x^n}{n!} + \frac{x^{n+1}}{(n+1)!} \tag{1}$$

   for $x > 0$; we will in fact show this for $x \geq 0$. Set $f(x) = e^x$ and $g(x) = 1 + x + \frac{x^2}{2} + \cdots + \frac{x^n}{n!} + \frac{x^{n+1}}{(n+1)!}$. Both functions are differentiable on $[0, \infty)$, and $f(0) = g(0) = 1$. We have $f'(x) = e^x$ and $g'(x) = 1 + x + \frac{x^2}{2} + \cdots + \frac{x^n}{n!}$. We have directly that $f'(0) = g'(0) = 1$; for $x > 0$ we have by induction that $f'(x) \geq g'(x)$. So, by our lemma, (1) holds.

   (b) Deduce

$$\lim_{x \to \infty} \frac{e^x}{x^n} = \infty.$$

   Solution: We have, for any $n \in \mathbb{N}$ and $x > 0$,

$$e^x \geq 1 + x + \frac{x^2}{2} + \cdots + \frac{x^n}{n!} + \frac{x^{n+1}}{(n+1)!} \geq \frac{x^{n+1}}{(n+1)!}$$

   so

$$\frac{e^x}{x^n} \geq \frac{x}{(n+1)!}.$$

   Since evidently $x/(n+1)! \to \infty$ as $x \to \infty$, we get that $e^x/n! \to \infty$ as $x \to \infty$.

8. *Newton's law of cooling* says that an object cools at a rate proportional to the difference between its temperature and the temperature of the surrounding medium. Suppose that an object has temperature T_0 at time $t = 0$, and that the temperature of the surrounding medium remains at a constant M throughout time.

   Find the temperature of the object at time t (in terms of T_0, M, and k, the implicit constant of proportionality in Newton's law).

Solution: Let $T(t)$ be the temperature at time t. Newton says that the function T satisfies the equation

$$T'(t) = -k(T(t) - M)$$

where k is some positive constant (the negative sign in the equation above indicates that the object is cooling).

To solve for T, consider the function $f(t) = T(t) - M$, which satisfies the equation

$$f'(t) = -kf(t).$$

One family of solutions to this equation is $f(t) = ce^{-kt}$, where c is some constant; that there are no other solutions follows from the same argument that we used to establish that the only solutions to $h'(x) = h(x)$ are $h(x) = ce^x$ (all this is assuming that T is a continuous function).

It follows that

$$T(t) = M + ce^{-kt}.$$

We have an initial condition: $T(0) = T_0$, from which we find that $T_0 = M + c$ or $c = T_0 - M$, leading to a solution

$$T(t) = M + (T_0 - M)e^{-kt}.$$