Math 10860, Honors Calculus 2

Homework 6 **NAME**:

Due in class Friday February 28

Instructions

Same as always.

Reading for this homework

Class notes Section 12.3 through 12.5.

Assignment

1. Consider the function $f(x) = \begin{cases} \frac{\sin x}{x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$.

   (a) Verify that f is continuous at 0. (We did this informally in the fall, now we can do it formally. This part should be trivial.)

   (b) Verify that f is differentiable at 0, and find $f'(0)$. (This requires a small ε-δ argument; it's the argument we skipped in class, when we talked about sin and cos being differentiable at $0, \pm\pi, \pm 2\pi$, et cetera. Basically what you have to show, either in general or for this specific example, that if f is defined on some open interval that contains a, f is continuous, f is differentiable everywhere except (possibly) at a, and if there is some number L such that f' approaches L near a, then in fact f is differentiable at a, and the derivative there is L).

2. Compute
$$\lim_{x \to 0} \left(\frac{1}{x} - \frac{1}{\sin x} \right).$$

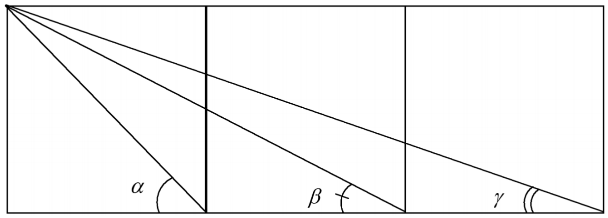
3. This question is about Machin's formula for π.

   (a) Prove that for all α and β for which all of $\tan(\alpha + \beta)$, $\tan \alpha$ and $\tan \beta$ exist,
$$\tan(\alpha + \beta) = \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta}.$$

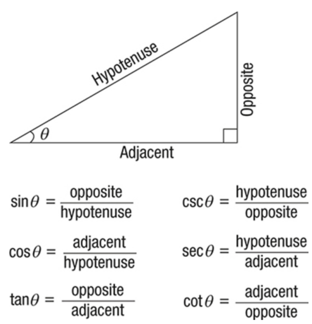
(b) Deduce from the previous part that for certain x, y,

$$\arctan x + \arctan y = \arctan\left(\frac{x+y}{1-xy}\right).$$

Determine exactly what conditions on x, y make this identity valid.



(c) (This part is an aside, but hopefully a cute one). The picture above shows a 1 by 3 rectangle divided into three 1 by 1 squares. Show that $\alpha = \beta + \gamma$. You may assume the connection between our analytic trigonometric functions, and the ratios of sides of right-angled triangles, as illustrated below:



(d) Prove Machin's formula:

$$\frac{\pi}{4} = 4\arctan\frac{1}{5} - \arctan\frac{1}{239}.$$

4. (a) Find formulae for $\sin 3x$ in terms of $\sin x$, and for $\cos 3x$ in terms of $\cos x$.

   (b) Deduce that (not unexpectedly) $\sin\frac{\pi}{6} = \frac{1}{2}$ and $\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$.

5. If $t = \tan(x/2)$ find simple expressions for $\sin x$ and $\cos x$ in terms of t. ("Simple" here means that the expressions should be rational functions in t. The only assumption you should make on x is that $\tan(x/2)$ is defined.)

6. This question is about the *hyperbolic trigonometric functions* sinh and cosh, defined as follows:

$$\begin{array}{lcl} \sinh : \mathbb{R} \to \mathbb{R} & \quad\text{and}\quad & \cosh : \mathbb{R} \to \mathbb{R} \\ x \mapsto \frac{e^x - e^{-x}}{2} & & x \mapsto \frac{e^x + e^{-x}}{2}. \end{array}$$

It is a long question, and I don't expect you to do all of it carefully. You *should* do every part in a less careful, scratch-work sort of way (in particular, you will do your soul good if you verify Part (h)), but here are the only parts that you should turn in for grading:

- Part (c)
- Parts (d)(iii) and (d)(vii)
- Parts (e), (f) and (g).

(a) Look at a graph of sinh. Check that all the features of the graph follow from basic calculus considerations: sinh is increasing from $-\infty$ (at $-\infty$) to ∞ (at ∞), never has zero derivative, is concave for negative x, convex for positive x, and passes through the origin.

(b) Look at a graph of cosh. Check that all the features of the graph follow from basic calculus considerations: cosh decreases from ∞ (at $-\infty$) to 1 (at 0), then increases to ∞ (at ∞), and is always convex.

(c) Name the famous monument, located in the midwest, whose shape is an upside-down cosh graph[1].

(d) Check that sinh and cosh satisfy the following identities, that are very reminiscent of identities satisfied by sin and cos:

i. $\cosh^2 - \sinh^2 = 1$.

ii. $\tanh^2 + 1/\cosh^2 = 1$ (Here tanh is defined to be $\sinh/\cosh$; note that its domain is all reals).

iii. $\sinh(x+y) = \sinh x \cosh y + \sinh y \cosh x$

iv. $\cosh(x+y) = \cosh x \cosh y + \sinh y \sinh x$.

v. $\sinh' = \cosh$.

vi. $\cosh' = \sinh$.

vii. $\tanh' = 1/\cosh^2$

(e) sinh is invertible, with inverse $\sinh^{-1} : \mathbb{R} \to \mathbb{R}$. Because sinh never has zero derivative, $\sinh^{-1}$ is differentiable everywhere. Find a very simple expression for $(\sinh^{-1})'(x)$ (one that does not involve hyperbolic trigonometric functions).

(f) Verify that $\sinh^{-1}(x) = \log(x + \sqrt{x^2+1})$ for all real x.

(g) Calculate $\int_a^b \frac{dt}{\sqrt{1+t^2}}$.

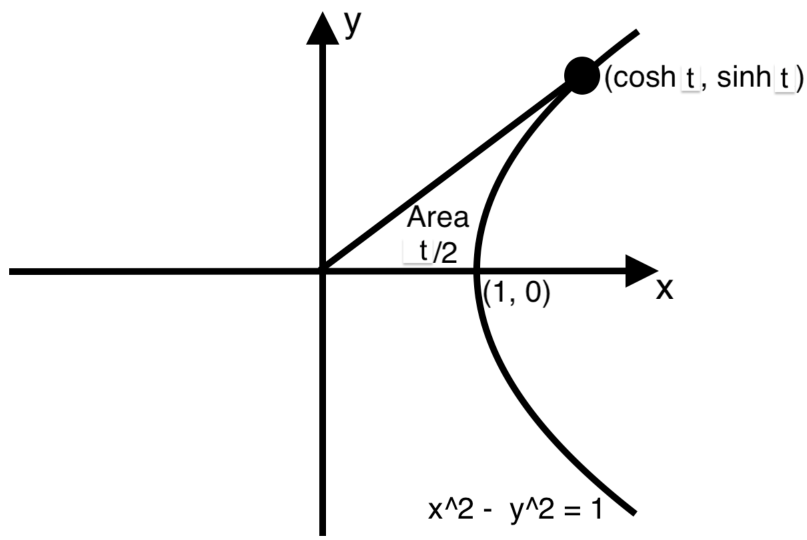
(h) This question justifies the name "hyperbolic trigonometric functions" for sinh and cosh.

Consider the curve in the (x,y)-plane consisting of all points (x,y) satisfying $x^2 - y^2 = 1$ (this curve is called a *hyperbola*). Let $P = (a,b)$ be a point on the curve, with $a \geq 1$ and $b \geq 0$. Suppose that the area A bounded by

- the x-axis between $(1,0)$ and $(0,0)$,
- the line segment from $(0,0)$ to P, and
- the curve $x^2 - y^2 = 1$ between P and $(1,0)$

[1] This is not some math professor BS. See, for example, the "Mathematical elements" section of its Wikipedia page. (You can also find the verification on its official page, but Wikipedia presents it better.)

is $t/2$ (see the picture below). Prove that $a = \cosh t$ and $b = \sinh t$. (So: the *hyperbolic* trigonometric functions can be defined in **exact** analogy with the ordinary trigonometric functions, by replacing the circle $x^2 + y^2 = 1$ with the *hyperbola* $x^2 - y^2 = 1$).



An extra credit problem: Prove that $(\sin x)^{\sin x} < (\cos x)^{\cos x}$ for all $x \in (0, \pi/4)$.